MA 237-02 §3.1 – 6.1	**Test #2**	*score*	**Name:** ____________ **19 November 2001**

INSTRUCTIONS: Answers to question 7 may be written on this page. All other problems should be worked on a separate sheet.

1. Find a matrix that induces a transformation from $\mathbb{R}^2$ to itself that sends the standard unit square in the first quadrant (with vertices (0,0), (1,0), (1,1), and (0,1)) to the parallelogram with vertices (0,0), (2,-1), (1,-2), and (-1,-1). How many such matrices are possible? *(10 points)*

   Solution: There are two such matrices obtained by examining where the standard basis elements get sent:

$$\begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix}$$

2. Give an example of two matrices A and B such that $AB \neq BA$, if such an example exists. *(10 points)*

   Solution: Matrix multiplicaiton is not, in general, commutative. You can choose a pair of 2×2 matrices to illustruate this, e.g., let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

3. Use the augmented matrix method to find (by hand) the inverse of the following matrix. *(10 points)*

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

   Solution: The inverse of the given matrix is

$$\begin{bmatrix} -3 & 1 & 1 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

4. Two $n \times n$ matrices A and B are called *similar* if there exists an invertible matrix Q such that $A = QBQ^{-1}$. Prove that similar matrices always have the same determinant. *(10 points)*

   Solution: $\det(A) = \det(QBQ^{-1}) = \det(Q)\det(B)\det(Q^{-1}) = \det(Q)\det(Q^{-1})\det(B) = \det(Q)\frac{1}{\det(Q)}\det(B) = \det(B)$

5. Find the projection of the vector $[1,2,3]^t \in \mathbb{R}^3$ onto the subspace of $\mathbb{R}^3$ spanned by the two vectors $[2,1,2]^t$ and $[1,2,1]^t$. *(10 points)*

   Solution: Let $\mathcal{W} = \text{span}(W_1, W_2)$ where W_1 and W_2 are the two given spanning vectors. Using the Gram-Schmidt method, we find an orthogonal basis $\{Q_1, Q_2\}$ for $\mathcal{W}$ by setting $Q_1 = W_1$ and $Q_2 = W_1 - \frac{W_2 \cdot W_1}{W_1 \cdot W_1} \cdot W_1 = [-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}]^t$. Then we calculate $\text{proj}_{\mathcal{W}} X = \frac{X \cdot Q_1}{Q_1 \cdot Q_1} \cdot Q_1 + \frac{X \cdot Q_2}{Q_2 \cdot Q_2} \cdot Q_2 = [2,2,2]^t$

6. Show that the vector X is an eigenvector for the matrix A and determine the corresponding eigenvalue. *(10 points)*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 5 \\ 8 \\ 7 \end{bmatrix}$$

Solution: A check shows that $AX = 4X$, so X is an eigenvector with corresponding eigenvalue 4.

7. For each of the following, answer TRUE if the given statement in always true. Otherwise, answer FALSE. *(5 points each)*

   (a) For the vector $[1,2,3]^t \in \mathbb{R}^3$, its coordinates in the basis $[2,1,0]^t$, $[1,0,4]^t$, $[1,-1,0]^t$ are $[1,1,-2]^t$.

   Solution: A check shows that
   $$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$
   so the statement is FALSE.

   (b) No linear transformations from $\mathbb{R}^4$ to $\mathbb{R}^3$ are one-to-one. ___________

   Solution: TRUE since the nullspace for any such transformation has dimension 1 or larger.

   (c) All linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^4$ are one-to-one. ___________

   Solution: FALSE, since the zero transformation (induced by the 4×3 matrix of zeros) is not one-to-one.

   (d) A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ is one-to-one if and only if it is onto. ___________

   Solution: TRUE by the Inverse Theorem on page 167 of the text.

   (e) If A' is obtained from a square matrix A by replacing all of the entries of A by their negatives, then $\det(A') = -\det(A)$. ___________

   Solution: FALSE; since if A has an odd number of rows $\det(A') = -\det A$. The correct general formula would be $\det(A') = (-1)^n \det A$ where n is the dimension of A.

   (f) For an invertible matrix A, $\det(A^{-1}) = \dfrac{1}{\det(A)}$. ___________

   Solution: TRUE by assigned Exercise 12 on page 288.

8. Short Answer *(5 points each)*

   (a) Suppose A is a 5×5 matrix and that the dimension of the nullspace of A is 2. Find the dimension of the image of the transformation. Briefly explain.

   Solution: The rank of A is 3 $(5-2)$, and that is the dimension of the image of the transformation.

   (b) Let A be a 3×4 matrix and B a 4×3 matrix. For the transformations determined by the matrix products AB and BA, describe whether or not either can be one-to-one or onto.

   Solution: The product AB is a 3×3 matrix that will induce a transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$. As such, AB might be both 1-1 and onto (for example, if AB is the identity matrix), or AB might be neither 1-1 nor onto. Those are the only two possibilities. BA, on the other hand, is a 4×4 matrix that will induce a transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$. From the diagram below, you see that the induced transformation cannot be 1-1 since it begins by going down a dimension. Thus T_{BA} cannot be onto either since such transformation are 1-1 if and only they are onto.

$$\begin{tikzcd} & \mathbb{R}^4 \arrow[dr, "T_A"] & \\ \mathbb{R}^3 \arrow[ur, "T_B"] \arrow[rr, "T_{AB}"] & & \mathbb{R}^3 \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{R}^3 \arrow[dr, "T_B"] & \\ \mathbb{R}^4 \arrow[ur, "T_A"] \arrow[rr, "T_{BA}"] & & \mathbb{R}^4 \end{tikzcd}$$