Summary of Definitions and Theorems

MA 237 – Linear Algebra

1 Systems of Linear Equations

1.1 Vectors

Definition 1 In $\mathbb{R}^2$, if $A = (x_1, y_1)$ and $B = (x_2, y_2)$, the the *dot product* of A and B is $A \cdot B = x_1x_2 + y_1y_2$.

Theorem 1 (page 10) *Two vectors A and B in $\mathbb{R}^2$ are perpendicular if and only if $A \cdot B = 0$.*

1.2 The Vector Space of $m \times n$ Matrices

Definition 2 (page 20) A non-empty set $\mathcal{V}$ that has operations of addition (+) and scalar multiplication ($\cdot$) defined on it is a *vector space* provided that

1. $\mathcal{V}$ is closed under addition;
2. addition is commutative, i.e., $A_1 + A_2 = A_2 + A_1$ for every $A_1, A_2 \in \mathcal{V}$;
3. addition is associative, i.e., $(A_1 + A_2) + A_3 = A_1 + (A_2 + A_3)$ for every $A_1, A_2, A_3 \in \mathcal{V}$;
4. $\mathcal{V}$ possesses an additive identity element, denoted $\mathbf{0}$, with the property that $A + \mathbf{0} = \mathbf{A}$ for every $A \in \mathcal{V}$;
5. every element in $\mathcal{V}$ has an additive inverse, i.e., for any $A \in \mathcal{V}$, there exists an element denoted $-A$ with the property that $A + (-A) = \mathbf{0}$;
6. $\mathcal{V}$ is closed under scalar multiplication, i.e., for every $k \in \mathbb{R}$ and every $A \in \mathcal{V}$, $k \cdot A \in \mathcal{V}$;
7. $(k \cdot l) \cdot A = k \cdot (l \cdot A)$ for every $k, l \in \mathbb{R}$ and every $A \in \mathcal{V}$;
8. $k \cdot (A_1 + A_2) = k \cdot A_1 + c \cdot A_2$ for every $k \in \mathbb{R}$ and every $A_1, A_2 \in \mathcal{V}$;
9. $(k + l) \cdot A = k \cdot A + l \cdot A$ for every $k, l \in \mathbb{R}$ and $A \in \mathcal{V}$;
10. $1 \cdot A = A$ for every $A \in \mathcal{V}$.

The main examples of vector spaces we have are $\mathbb{R}^n$ and $M(m, n)$, the space of $m \times n$ matrices.

Definition 3 (pages 17-18) A set of vectors $\{A_1, A_2, \ldots, A_n\}$ is *dependent* if at least one of the vectors can be expressed as a linear combination of the others. Otherwise, the set is *independent*.

Definition 4 (page 19) Let $\mathcal{V}$ be a vector space and let $\{A_1, A_2, \ldots, A_n\} \subseteq \mathcal{V}$. Then the *span* of the set $\{A_1, A_2, \ldots, A_n\} \subseteq \mathcal{V}$ is the set of all linear combinations of the vectors, i.e.,

$$\text{span}(\{A_1, A_2, \ldots, A_n\}) = \{c_1 \cdot A_1 + c_2 \cdot A_2 + \cdots + c_n \cdot A_n \mid c_i \in \mathbb{R} \text{ for all } i\}$$

1.3 Systems of Linear Equations

The process of Gaussian elimination was described; representing solutions of linear systems parametrically using a translation vector and spanning vectors was discussed.

1.4 Gaussian Elimination

Elementary row operations were described, echelon form and reduced echelon form were defined. Row equivalence of matrices was defined.

Theorem 2 (page 45) *Every matrix is row equivalent to a matrix in echelon form.*

Theorem 3 (More Unknowns Theorem, page 49) *A system of linear equations with more unknowns than equations will either fail to have any solutions or will have an infinite number of solutions.*

1.5 Column Space and Nullspace

Definition 5 (page 59) The *column space* of a matrix A is the span of the columns of A (thought of as individual column vectors).

Theorem 4 (page 59) *A linear system is solvable if and only if the vector of constants belongs to the column space of the coefficient matrix.*

Definition 6 (page 60) If $\mathcal{V}$ is a vector space and $\emptyset \neq \mathcal{W} \subseteq \mathcal{V}$, $\mathcal{W}$ is a *subspace* of $\mathcal{V}$ if $\mathcal{W}$ is closed under all linear combinations, i.e., given any $X, Y \in \mathcal{W}$ and $k, l \in \mathbb{R}$, $k \cdot X + l \cdot Y \in \mathcal{W}$.

Note that the above definition is equivalent to saying that a non-empty subset $\mathcal{W}$ of a vector space $\mathcal{V}$ is a subspace if and only if $\mathcal{W}$ is a vector space itself.

Theorem 5 (Translation Theorem, page 63) *Let T be any solution to the system $AX = B$. Then Y satisfies this system if and only if $Y = T + Z$ where Z satisfies $AZ = \mathbf{0}$.*

Definition 7 (page 64) The *nullspace* of an $m \times n$ matrix A is the set of solutions to the system of homogeneous equations $AX = \mathbf{0}$ where X is an $n \times 1$ column vector of variables.

Theorem 6 (page 66) *A subspace of a vector space $\mathcal{V}$ is a vector space under the addition and scalar multiplication operations inherited from $\mathcal{V}$.*

2 Linear Independence and Dimension

2.1 The Test for Linear Independence

Theorem 7 (Test for Independence, page 82) *A set of vectors $\{A_1, A_2, \dots, A_n\}$ is independent if and only if the only solution to the equation $k_1A_1 + k_2A_2 + \cdots + k_nA_n = 0$ is the zero solution (i.e., each $k_i = 0$).*

Theorem 8 (page 87) *The pivot columns for a matrix A are the columns corresponding to the pivot variables in the system $\mathbf{A} \cdot \mathbf{X} = \mathbf{0}$. The pivot columns for A are independent and the nonpivot columns are linear combinations of them.*

2.2 Dimension

Definition 8 The *dimension* of a vector space $\mathcal{V}$ is the smallest number of elements necessary to span $\mathcal{V}$.

Definition 9 A *basis* for a vector space $\mathcal{V}$ is a set of linearly independent elements that spans $\mathcal{V}$.

Theorem 9 (page 94) *In an n-dimensional vector space, there can be at most n independent elements.*

Theorem 10 (page 96) *If a vector space is n-dimensional, then any set of n elements that spans the space must be independent.*

Theorem 11 (page 97) *If a vector space can be spanned by n independent elements, the it is n-dimensional.*

Theorem 12 (Dimension Theorem, page 97) *A vector space $\mathcal{V}$ is n-dimensional if and only if it has a basis containing n elements. In this case, all bases contain n elements.*

Theorem 13 (page 100) *In an n-dimensional vector space $\mathcal{V}$, any set of n linearly independent elements spans $\mathcal{V}$.*

Theorem 14 (page 100) *In an n-dimensional space $\mathcal{V}$, any set of n elements that spans $\mathcal{V}$ must be independend and any set of n independent elements must span $\mathcal{V}$.*

2.3 Applications to Systems

Definition 10 (page 122) The row space of a matrix A is the span of the rows of A.

Theorem 15 (page 123) *Let A and B be two row-equivalent matrices. Then A and B have the same row space.*

Theorem 16 (Non-Zero Tows Theorem, page 124) *The non-zero rows of any echelon form of a matrix A form a basis for the row space of A.*

Definition 11 (page 125) The *rank* of a matrix A is the dimension of the row space. It is computable as the number of non-zero rows in an exhelon form of the matrix.

Theorem 17 (page 126) *For any matrix A, the row space and the column space have the same diminsion. This common dimension is the rank of A.*

Theorem 18 (Rank-Nullity page 127) *For any matrix A, the rank of A plus the dimension of the nullspace of A is the total number of columns of A.*

Theorem 19 (page 128) *If a matrix A is placed in row-reduced echolon form, and the nullspace expressed as a span of the vectors corresponding to the free columns, then those spanning vectors are independent.*