MA 237-01 §1.1 – 2.3	**Test #1 Solutions**	Name: ______________ 27 February 2002

INSTRUCTIONS: Work the following problems on your own paper.

1. Consider the following system of equations.

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 1 \\ 3x_1 - 2x_2 - x_3 &= 2 \end{aligned}$$

Write out the augmented matrix for the system of equations. Then use row operations to transform the augmented matrix into reduced echelon form. Document each row operation you use, e.g., with notation like $R_2 \to R_2 - R_1$. Then write the solutions in parametric form and describe the set of solutions geometrically. *(16 points)*

Solution:

$$\begin{bmatrix} 2 & -1 & 1 & 1 \\ 3 & -2 & -1 & 2 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 5 & -1 \end{bmatrix}$$

So, the solution can be written in parametric form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$$

The solution set is a line in $\mathbb{R}^3$.

2. For the following system of equations

$$\begin{aligned} 3x_1 - 2x_2 - 2x_3 + x_4 + 13x_5 &= 15 \\ 3x_1 + 2x_2 - 10x_3 + 2x_4 + 2x_5 &= 0 \\ x_1 + 2x_2 - 6x_3 - 2x_4 - 6x_5 &= -8 \end{aligned}$$

suppose that the augmented matrix for this system has echelon form

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 2 & 2 \\ 0 & 1 & -2 & 0 & -3 & -4 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(a) Express the set of solutions in parametric form (i.e., as a sum of a translation vector plus linear combinations of other vectors). Describe the set of solutions geometrically. *(8 points)*

Solution: The reduced echelon matrix gives rise to the system

$$\begin{aligned} x_1 &= x_3 - 2x_5 + 2 \\ x_2 &= 2x_3 + 3x_5 - 4 \\ x_4 &= -x_5 + 1 \end{aligned}$$

This system can be written in vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \cdot \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

The solution set is a plane in $\mathbb{R}^5$.

(b) Regarding the original system of equations as a matrix equation in the form $AX = B$, find a basis for the null space of A and state its dimension. *(8 points)*

Solution: The nullspace is 2-dimensional and has a basis the two vectors

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(c) Find a basis for the row space of the coefficient matrix A and state its dimension. *(8 points)*

Solution: The row space is 3-dimensional and has as a basis the vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

You could write these vectors as row vectors, if you prefer.

(d) Find a basis for the column space of A and state its dimension. *(8 points)*

Solution: The column space is also 3-dimensional and has as a basis the vectors from the original matrix

$$\left\{ \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \right\}$$

3. For each of the following, write TRUE if the statement is always true. Otherwise, write FALSE. Give a brief explanation. *(4 points each)*

(a) It is possible for a system of linear equations with more unknowns that equations to have no solutions.

Solution: TRUE, the system could be inconsistent.

(b) The set of solutions of any system of linear equations in n unknowns forms a subspace of $\mathbb{R}^n$.

Solution: FALSE; this is true only when the system in homogeneous; otherwise, the set of solutions is a translate of a subspace.

(c) Any two vectors in $\mathbb{R}^4$ are independent.

Solution: This is FALSE, e.g., if one of the vectors is the zero vector or if one vector is a scalar multiple of the other.

(d) If A is a 5×3 matrix, the rank of A is less than or equal to 3 and the dimension of the nullspace is 2 or larger.

Solution: FALSE; the rank does have to be 3 or less, but the nullspace dimension plus the rank is 3, so the nullspace dimension could be less than 2.

(e) Any four vectors in $\mathbb{R}^3$ are dependent.

Solution: This is TRUE since $\mathbb{R}^3$ is 3-dimensional and thus cannot have an independent subset with more than 3 vectors.

(f) For any matrix A, the dimension of the row space equals the dimension of the column space.

Solution: This is TRUE by a theorem in §2.3.

(g) For any $m \times n$ matrix A, the rank of A equals n minus the dimension of the nullspace of A.

Solution: This is TRUE since $n = \text{rank}(A) + \dim(\text{null}(A))$.

(h) The dimension of the vector space of $m \times n$ matrices, $M(m, n)$, is $m \cdot n$.

Solution: TRUE; there is a basis consisting of matrices having a single 1 with all other entries 0; there are $m \times n$ different such basis elements.

(i) A homogeneous system of linear equations cannot be inconsistent.

Solution: TRUE, there is always the zero solution.

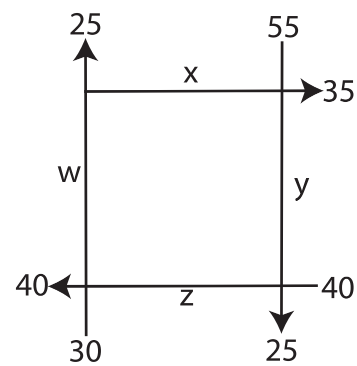
4. Construct a 3×4 matrix A so that the equation $AX = B$ has a solution if and only if $B \in \text{span}([2, 1, 3]^t)$. *(8 points)*

Solution: According to Theorem 1 in §1.5, we just need to construct a matrix having column space $\text{span}([2, 1, 3]^t)$. So a matrix like

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

will do the job.

5. The accompanying diagram shows the traffic flow (in cars per minute) on a collection of one-way streets. Find a system of equations that gives the relationships among the variables. You do not need to solve the system. *(8 points)*



Solution: At each vertex, just equate the input with the output to obtain

$$\begin{aligned} x + 55 &= y + 35 \\ y + 40 &= z + 25 \\ z + 30 &= w + 40 \\ w &= x + 25 \end{aligned}$$