MA 237-02 §3.1 – 6.1	**Test #2**	*score*	**Name:** ______________ **18 April 2002**

INSTRUCTIONS: Answers to question 8 may be written on this page. All other problems should be worked on a separate sheet.

1. Find the coordinates of the point $[-1,2]^t$ in the basis $\{[2,1]^t,[1,1]^t\}$ for $\mathbb{R}^2$. Show how you do this. *(10 points)*

   Solution: Form the matrix $Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and calculate $Q^{-1}X = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$.

2. Give an example of a 2×3 matrix A so that the image of the induced transformation of A consists of the line $y = 2x$ in the plane. State the domain, range, dimension of the image, and dimension of the null space for such a transformation? Briefly explain. *(10 points)*

   Solution: Since the range of a matrix transformation is equal to the span of the columns of the matrix, we just choose a 2×3 matrix in which the columns are all in the desired subspace ($y = 2x$). Examples, of such matrices are $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$, and $\begin{bmatrix} 1 & -1 & 3 \\ 2 & -4 & 6 \end{bmatrix}$. The domain of any such transformation is $\mathbb{R}^3$, the dimension of the range or image is 1, and the dimension of the nullspace is 2.

3. Give an example of two matrices A and B such that $(AB)^t \neq A^tB^t$ (show this), or state that such an example can't occur. *(10 points)*

   Solution: You can pick nearly any pair of 2×2 matrices to show this.

4. Use the augmented matrix method to find (by hand) the inverse of the following matrix. *(10 points)*

$$\begin{bmatrix} -1 & 3 & 1 \\ -1 & 2 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

   Solution:

$$\begin{bmatrix} -1 & 3 & 1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & -1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -3 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & -3 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & -3 & 1 & -1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 2 & -3 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 4 & -6 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & -3 & 1 \end{bmatrix}$$

   So, $\begin{bmatrix} -1 & 3 & 1 \\ -1 & 2 & 1 \\ -1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & -6 & 1 \\ 1 & -1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$.

5. Use the Gram-Schmidt process to convert the ordered basis $\{[1,1,1]^t,[2,1,2]^t,[1,-1,-1]^t\}$ into an orthogonal basis. Show your work. *(10 points)*

   Solution: To begin the process, let $A_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ Set $Q_1 = A_1$. Then calculate $Q_2 = A_2 -$

$\frac{A_2 \cdot Q_1}{Q_1 \cdot Q_1} \cdot Q_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$. For convenience, replace Q_2 with a parallel vector without the fractions: $Q_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Finally, calculate $Q_3 = A_3 - \frac{A_3 \cdot Q_1}{Q_1 \cdot Q_1} \cdot Q_1 - \frac{A_3 \cdot Q_2}{Q_2 \cdot Q_2} \cdot Q_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

6. Calculate the characteristic polynomial for the given matrix and determine all of the eigenvalues. Show your work. You do not need to find any eigenvectors on this problem. *(10 points)*

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution: Compute the following determinant using any method, e.g., expanding along the first row.

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 2 & 1-\lambda \\ 1 & 2 \end{vmatrix} =$$

$(1-\lambda)((1-\lambda)^2 - 4) - (2(1-\lambda) - 2) + (4 - (1-\lambda)) = (1-\lambda)(-3 - 2\lambda + \lambda^2) - (-2\lambda) + (3+\lambda) =$

$-\lambda^3 + 3\lambda^2 + 4\lambda$

7. Since the matrix below is in triangular form, you know that $\lambda = 2$ is an eigenvalue for the matrix. Determine the corresponding eigenspace by exhibiting an eigenvector (or collection of independent eigenvectors) that span(s) the eigenspace. Show your work. *(10 points)*

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

Solution: Solve the homogeneous system represented by the matrix for $\lambda = 2$. $\begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & -1-\lambda \end{bmatrix} =$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -3 \end{bmatrix}$. This last matrix contains only the information that $x_1 + x_2 - 3x_3 = 0$, so $X = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

So the eigenspace corresponding to $\lambda = 2$ is the span of the set $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

8. For each of the following, answer TRUE if the given statement in always true. Otherwise, answer FALSE. *(3 points each)*

   (a) Any subspace of $\mathbb{R}^n$ has an orthogonal basis.

   Solution: TRUE; just take any basis and apply the Gram-Schmidt process.

   (b) For any invertible matrix A, $|A^{-1}| = \dfrac{1}{|A|}$.

   Solution: TRUE, since $AA^{-1} = I$ so $|A| \cdot |A^{-1}| = 1$.

(c) For any square matrices A and B of the same size, $|AB| = |BA|$.

Solution: TRUE, since $|AB| = |A| \cdot |B| = |B| \cdot |A| = |BA|$.

(d) If a matrix B is obtained from an $n \times n$ matrix A by interchanging exactly two rows, then $|A| = |B|$.

Solution: FALSE. Row interchanges negate the determinant.

(e) Any linear transformation from $\mathbb{R}^1$ to $\mathbb{R}^2$ is one-to-one.

Solution: FALSE, for example the transformation that sends everything to 0 is not 1-1.

(f) A square matrix with two identical columns has a determinant of 0.

Solution: TRUE, since one column operation will yield a column of zeros.

(g) A square matrix is invertible if and only if the associated linear transformation is onto.

Solution: TRUE. This is one of our many equivalences to invertibility.

(h) If A is a square matrix, and if $AX = B$ has no solutions for some vector B, then A is not invertible.

Solution: TRUE, since such a vector B would not be in the column space of A, so A would not be onto.

(i) If A is a 5×3 matrix and B is a 3×4, the transformation induced by the product matrix AB is never one-to-one.

Solution: TRUE, since B is a transformation from $\mathbb{R}^4 \to \mathbb{R}^3$, it is not 1-1. Following it by A will not change this.

(j) If A is a 5×3 matrix and B is a 3×4, the transformation induced by the product matrix AB is never onto.

Solution: TRUE. AB is a transformatino from $\mathbb{R}^4 \to \mathbb{R}^5$, so AB can't be onto.