Inclusion-Exclusion and Special Numbers

1 Introduction

There are different ways to state the inclusion-exclusion principle. One which is different from that in the text but which may be easier to follow is

Theorem 1 (Inclusion-Exclusion Principle). *For any collection* $\{A_i\}_{i=1}^{n}$ *of sets*

$$\begin{aligned}\left|\bigcup_{i=1}^{n} A_i\right| &= \sum_{i=1}^{n}|A_i| - \sum_{1\le i_1<i_2\le n}|A_{i_1}\cap A_{i_2}| + \sum_{1\le i_1<i_2<i_3\le n}|A_{i_1}\cap A_{i_2}\cap A_{i_3}| - \cdots \\ &\quad + (-1)^{n+1}|A_1\cap A_2\cap\cdots\cap A_n| \\ &= \sum_{\substack{1\le j\le n \\ 1\le i_1<i_2<\cdots<i_j\le n}} (-1)^{j+1}\left|\bigcap_{k=1}^{j} A_{i_k}\right|\end{aligned}$$

Proof. Let $x \in \bigcup_{i=1}^{n} A_i$. Suppose x is in exactly k of the sets $\{A_i\}$. Then we can count the number of times x gets counted on the right-hand side of the equation by

$$\begin{aligned}&\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \cdots (-1)^{n+1}\binom{k}{n} \\ &= \binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \cdots (-1)^{k+1}\binom{k}{k} \\ &= 1 - (1-1)^k \\ &= 1\end{aligned}$$

Thus each element counted on the left side gets counted exactly once on the right. □

Exercise 1: The second summation sign on the right-hand-side of the inclusion-exclusion equation involves an iteration that can be described by the following nested `for` loops. How many times will the statement be printed in the following program?

```
for i = 1 to n-1;
  for j = i+1 to n;
    print("Hello class");
  next j;
next i;
```

Exercise 2: The third summation sign on the right-hand-side of the inclusion-exclusion equation involves an iteration that can be described by the following three nested `for` loops. How many times will the statement be printed in the following program?

```
for i = 1 to n-2;
  for j = i+1 to n-1;
    for k = j+1 to n;
      print("Hello again class");
    next k;
  next j;
next i;
```

2 Onto Functions

Let $\mathbb{N}_n = \{1, 2, \ldots, n\}$. The basic question for us here is "how many functions $f : \mathbb{N}_m \to \mathbb{N}_n$ are onto?" We have looked at special cases of this question already. The general case can be resolved by inclusion-exclusion.

Let A_i = the set of functions from $\mathbb{N}_m \to \mathbb{N}_m$ which omit the number i from their range. Then

$$\begin{aligned} |A_i| &= (n-1)^m \\ \left|A_i \cap A_j\right| &= (n-2)^m \qquad \text{(if } i \neq j) \\ & \qquad\qquad\qquad\qquad \text{etc.} \end{aligned}$$

So,

$$\begin{aligned} \left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}\right| &= \left|\overline{A_1 \cup A_2 \cup \cdots \cup A_n}\right| \\ &= n^m - \left[n(n-1)^m - \binom{n}{2}(n-2)^m + \cdots + (-1)^{n+1}\binom{n}{n-1}1^m\right] \\ &= \sum_{i=0}^{n-1}(-1)^i\binom{n}{i}(n-i)^m \end{aligned}$$

3 Stirling Numbers of the Second Kind

The number of surjective (onto) functions from $\mathbb{N}_m \to \mathbb{N}_n$ is closely related to one of the types of Stirling numbers. Stirling studied two important classes of numbers that occur as coefficients in special polynomials. The second type he studied occurs more naturally in this context. We will discuss the first type later. The *Stirling numbers of the second kind*, $S(m, n)$, are defined to be the number of ways to distribute m distinguishable objects into n identical boxes with no box left empty. Equivalently, $S(m, n)$ is the number of partitions of a set of size m into n non-empty (disjoint) subsets. In order to understand this last statement, we need some definitions.

Definition 1. A *partition* of a non-empty set S is a collection of non-empty subsets of S, $\mathcal{A} = \{A_1, \ldots, A_n\}$ so that $\cup_{i=1}^n A_i = S$ and the collection $\mathcal{A}$ is pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Definition 2. For any function $f : A \to B$, the notation $f^{-1}(b)$ denotes $\{a \in A | f(a) = b\}$. The set $f^{-1}(b)$ is called the set of *pre-images* of b under f.

Note that for any onto function $f : A \to B$, the collection $\{f^{-1}(b) | b \in B\}$ is a partition of A into $|B|$ sets. Also note that, given any partition of A into $|B|$ sets, there are $|B|!$ ways to define a function from A to B so the pre-images match the partition sets.

Since $S(m,n) = \frac{1}{n!}$(number of onto functions from $\mathbb{N}_m \to \mathbb{N}_n$), we have a formula for Stirling numbers of the second kind:

$$S(m,n) = \frac{1}{n!} \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^m \tag{1}$$

EXERCISE 3: Show that $S(m,n) = S(m-1,n-1) + nS(m-1,n)$, i.e., devise a combinatorial argument that shows this equation holds in general. Then make a table of values for $1 \le m \le 8$ and $1 \le n \le m$.

4 Derangements

Of the $n!$ permutations of the elements in the set $\mathbb{N}_n$, some have an integer i in the i^{th} position (for some i) and some do not have any numbers in their correct position. We call those that do not *derangements*. The number of derangements of a set of size n can be computed using inclusion-exclution on the complementary event.

EXERCISE 4: Find a formula for the number of derangements, D_n, of a set with n elements.

EXERCISE 5: A hat check person discovers that n people's hats have been mixed up and returns these hats to the owners at random.

1. In how many ways can the hats be returned so that all of the owners get the wrong hat?
2. What proportion of the total number of distributions do these represent?
3. What limiting value does this proportion have (as $n \to \infty$)?

EXERCISE 6: Show that

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \qquad \text{for } n \ge 3 \tag{2}$$

$$D_n = nD_{n-1} + (-1)^n \qquad \text{for } n \ge 2 \tag{3}$$

5 Combinations with Repetition

For convenience, we introduce the notion of a multiset.

Definition 3. A *multiset* is an ordered pair (S, f) where S is a non-empty set and $f : S \to \mathbb{N}$.

What the heck does that mean? Well, if we want to count the number of ways to select three marbles from five red ones and four green ones, we could phrase the problem as "how many 3-combinations are there from the multiset $\{5r, 4g\}$?" Here, the underlying set is $\{r, g\}$ and the function is just $f(r) = 5$ and $f(g) = 4$. So multisets just provide a convenient notation/terminology for counting problems in which there is a limited supply of objects.

We already have formulæ for the number of r-combinations of n distinct objects and also for the number of r-combinations of a multiset with k distinct objects each with infinite repetition number. This latter count can be modelled by placing r identical balls in k distinct boxes so the number is $\binom{k+r-1}{r}$.

Using inclusion-exclusion, we can now determine the number of r-combinations of a multiset where the repetition numbers are not infinite. In other words, we can count the number of ways r identical balls can be placed in k distinct boxes where we require that each box gets no more than some particular number of balls. Before when we encountered problems of this type, they we contrived so that they were not too complicated. Now we can handle the general case of such problems.

EXAMPLE 1: Find the number of 10-combinations of the multiset $\{5a, 4b, 3c\}$.

Solution: We can think of placing 10 identical balls into three distinct boxes where the first box gets 5 or fewer balls, the second gets 4 or fewer, etc. Let A be the set of distributions of 10 balls into the three boxes so that the first box has more than 5 balls, let B be the set of distributions so that the second box has more than 4 balls, etc. Then

$$\begin{aligned}
\left|\overline{A} \cap \overline{B} \cap \overline{C}\right| &= \left|\overline{A \cup B \cup C}\right| \\
&= \binom{10+3-1}{10} - |A \cup B \cup C| \\
&= \binom{12}{10} - [|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|] \\
&= \binom{12}{10} - \left[\binom{6}{4} + \binom{7}{5} + \binom{8}{6} - 0 - 1 - 3 + 0\right] \\
&= 66 - 15 - 21 - 28 + 1 + 3 \\
&= 6
\end{aligned}$$

EXAMPLE 2: Find the number of solutions to the equation

$$x_1 + x_2 + x_3 = 12$$

subject to the constraints

$$\begin{aligned}
1 \le x_1 \le 5 \\
1 \le x_2 \le 4 \\
1 \le x_3 \le 3
\end{aligned}$$

Solution: This is the same problem as the previous one.

EXERCISE 7: Use inclusion-exclusion to find the number of integers between 1 and 10,000 inclusive which are not divisible by 4, 5, or 6.

EXERCISE 8: Determine the number of solutions of the equation $x_1 + x_2 + x_3 = 14$ in positive integers not exceeding 8. Do the same problem for non-negative integers not exceeding 8.

EXERCISE 9: In how many ways can eight pieces of identical candy be passed out to three children so that each child gets at least one piece but no child gets more than four pieces?

6 Stirling Numbers of the First Kind

Recall our notation for r-permutations of n objects: $P(m,n) = m(m-1)\cdots(m-n+1)$.

If we regard $P(m,n)$ as a polynomial in the formal variable x, we get $P(x,n) = x(x-1)\cdots(x-n+1)$ where m has been replaced by x to emphasize that we are regarding this as a polynomial in (now) x. Expanding, we get a polynomial of degree n of the form $P(x,n) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$.

Definition 4. The *Stirling numbers of the first kind* are defined by $s(n,j) = a_j$ where the a_j's are as above. In other words, the Stirling numbers of the first kind, $s(n,j)$, are the polynomial coefficients determinied by the equation

$$P(x,n) = x(x-1)\cdots(x-n+1) = \sum_{j=1}^{n} s(n,j)x^j. \tag{4}$$

EXERCISE 10: Write $P(x,3)$ as a polynomial in standard form and read off the $s(3,j)$ values. Now calculate the $s(4,j)$ values similarly.

The Stirling numbers of the first kind satisfy a recurrence relation as follows.

Theorem 2. *For any integers $m,n \geq 0$,*

$$s(m,n) = s(m-1,n-1) - (m-1)s(m-1,n). \tag{5}$$

Proof. Note that

$$\begin{aligned}
\sum_{j=0}^{m} s(m,j)x^j &= P(x,m) \\
&= P(x,m-1)\cdot(x-m-1) \\
&= \sum_{i=0}^{m} s(m-1,i)\cdot(x-m+1) \\
&= \sum_{i=0}^{m-1} \Big[s(m-1,i)\, x^{i+1} - (m-1)\, s(m-1,i)\, x^i \Big].
\end{aligned}$$

Equating coefficients of like terms on the two sides gives the result. □

EXERCISE 11: Construct a triangle (table) for the Stirling numbers of the first kind.

Recall that Stirling numbers of the second kind, $S(m,n)$, were defined earlier as the number of set partitions of a set of size n into exactly m non-empty subsets. It turns out that these Stirling numbers can also be view as polynomial coefficients.

Theorem 3. *For any positive integers, m and n,*

$$n^m = \sum_{j=0}^{n} S(m,j)\,P(n,j).$$

Proof. The proof relies on the following three observations:

1. The total number of functions $f : \mathbb{N}_m \to \mathbb{N}_n$ is n^m.
2. Each function $f : \mathbb{N}_m \to \mathbb{N}_n$ is onto some subset of $\mathbb{N}_n$.
3. There are $\binom{n}{k}$ of size k in $\mathbb{N}_n$.

Putting these observations together, we have

$$\begin{aligned} n^m &= \sum_{j=0}^{n} \binom{n}{j} S(m,j)\, j! \\ &= \sum_{j=0}^{n} \frac{n!}{(n-j)!} S(m,j) \\ &= \sum_{j=0}^{n} S(m,j)\,P(n,j) \end{aligned}$$

□

Corollary 1. *The Stirling numbers $S(n,j)$ can be used to re-write the monomial x^n as a linear combination of falling factorials*

$$x^n = \sum_{j=0}^{n} S(m,j)\,P(x,j).$$

EXERCISE 12: Write x^3 and x^4 as a sum of falling factorials, i.e., as a linear combination of terms of the form $P(x,j)$.

7 Eulerian Numbers

Recall that the binomial coefficients $\binom{n}{k}$ satisfy the property that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$. In terms of Pascal's triangle, this says that the sum of the numbers on the n^{th} row is 2^n. In this section we will examine a different collection of numbers that have properties similar to the binomial coefficients. They will be symmetric, they will fit naturally in a triangle, but rows in the triangle will sum to $n!$.

Definition 5. The *Eulerian numbers*, denoted $\left\langle {n \atop k} \right\rangle$, are defined as the number of permutations on $\mathbb{N}_n$ that have exactly k rises. (A rise occurs when a number in a permutation is larger than its predecessor.)

Some authors define Euler numbers slightly differently than the above definition, so you always have to check the definition being used if you encounter Euler numbers elsewhere.

It is easy to construct a combinatorial argument that shows the Euler numbers satisfy a recurrence relation of the form

$$\left\langle {n \atop k} \right\rangle = (n-k) \left\langle {n-1 \atop k-1} \right\rangle + (k+1) \left\langle {n-1 \atop k} \right\rangle, \ n \geq 0, \ k \in \mathbb{N}, \tag{6}$$
$$\left\langle {0 \atop 0} \right\rangle = 1, \quad \left\langle {0 \atop k} \right\rangle = 0 \ (k \neq 0)$$

EXERCISE 13: Use the recurrence relation to make a table of Eulerian numbers, either as a right triangle (table), or as an equilateral triangle (as is common in Pascal's triangle). Check the sum on each row and see what you get. Do you see a pattern?

EXERCISE 14: List the 24 permutations of $\mathbb{N}_4$ (you may want to begin with $\mathbb{N}_3$ as a warmup). For each permutation, count the number of rises, i.e., count the number of times a given number is larger than its predecessor. For example, the permutation 3412 has the form $3 \uparrow 4 \downarrow 1 \uparrow 2$ and has 2 rises. Now total the number of permutations that have 0 rises, then 1 rise, etc. Compare the numbers you get to the fourth row in the Eulerian triangle computed in the previous exercise. Make a conjecture.

Euler numbers are symmetric since the number of permutations with k rises equals the number of k falls. This observation gives

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle \tag{7}$$

Euler numbers occur in solutions to several mathematical and statistical problems. One well-known problem is the "Smith College diploma problem" described in [**?**]. They also occur in several combinatorial formulæ. For example, the Euler numbers are the coefficients needed to express the monomial x^n as a linear combination of binomial coefficients:

$$x^n = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle \binom{x+k}{n}. \tag{8}$$

The Euler numbers also can be expressed in a closed-form summation formula similar to that of the Stirling numbers of the second kind.

$$\left\langle {n \atop m} \right\rangle = \sum_{k=0}^{m} \binom{n+1}{k} (m+1-k)^n (-1)^k \tag{9}$$

EXERCISE 15: In [**?**], the following problem is posed. The notation has been adjusted to match this handout. Show that

$$\sum_{j=0}^{n-1} 2^j \left\langle {n \atop j} \right\rangle = \sum_{j=1}^{n} S(n,j)\, j!.$$

References

[1] R.L. Graham, D.E. Knuth, and O. Patashnik, *Concrete mathematics*, Second Edition, Addison-Wesley, Reading, MA, 1994.

[2] R. Maurer, *Problem E2404*, Amer. Math. Monthly **80** (1973), 316.

[3] Western Maryland College Problems Group, *Problem 11007*, Amer. Math. Monthly **110** (4), April 2003, 340.