MA 367-01 **§5.5 – 6.1**	# Quiz #3	*score*	**Name:** ____________ **10 February 2003**

INSTRUCTIONS: Turn in solutions to the following problems by Friday (14 February 2003) in class. As usual, fully explain your solutions and calculate the numerical values (when appropriate).

1. Show that
$$\left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}\right]^2=\sum_{k=0}^{2n}\binom{2n}{k}.$$

Solution: There are several ways to verify this identity. Here's one.

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k} &= 2^n && \text{(basic identity)}\\
\left[\sum_{k=0}^{n}\binom{n}{k}\right]^2 &= (2^n)^2 && \text{(square both sides)}\\
\left[\sum_{k=0}^{n}\binom{n}{k}\right]^2 &= 2^{2n} &&\\
\left[\sum_{k=0}^{n}\binom{n}{k}\right]^2 &= \sum_{k=0}^{2n}\binom{2n}{k} && \text{(basic identity)}
\end{aligned}$$

Here's another, described briefly. Every walk from row 0 in Pascal's triangle to row $2n$ can be uniquely split into a walk from row 0 to a point on row n followed by a walk from that point on row n to row $2n$. The result follows using the formula for the number of such walks. Can you supply the details for this?

2. Evaluate
$$\sum_{k=1}^{n}(-1)^k k\binom{n}{k}.$$

Solution: Begin with $\sum_{k=0}^{n}\binom{n}{k}x^k=(1+x)^n$. Differentiate both sides to obtain $\sum_{k=1}^{n}k\cdot\binom{n}{k}x^{k-1}=n\cdot(1+x)^{n-1}$. Substituting $x=-1$, we obtain $\sum_{k=1}^{n}(-1)^{k-1}k\cdot\binom{n}{k}=n\cdot(1-1)^{n-1}$, and multiplying both sides by -1 yields $\sum_{k=1}^{n}(-1)^k k\cdot\binom{n}{k}=-n\cdot 0^{n-1}$. This last expression equals 0 if $n>1$, but is undefined (0^0) when $n=1$. A separate evaluation for $n=1$ shows the sum is -1.

3. Enumerate all the permutations of the letters a, b, c, d

   (a) in lexicographic order.

   (b) Since that was so much fun, do it again in a minimum change order this time, i.e., so that each permutation is obtained from its predecessor by interchanging a single pair of letters. Better yet, make the last permutation in the list differ from the first by a single interchange, also.

Solution: There is only one way to do the enumeration lexicographically, but there are several ways to do the minimum change enumeration.

Lexicographic	Minimum Change
abcd	abcd
abdc	abdc
acbd	adbc
acdb	adcb
adbc	acdb
adcb	acbd
bacd	dcba
badc	dcab
bcad	dacb
bcda	dabc
bdca	dbac
bdac	dbca
cabd	cbda
cadb	cbad
cbad	cabd
cdba	cadb
cdab	cdab
cdba	cdba
dabc	bdca
dabc	bdac
dbac	badc
dbca	bcda
dcab	bcad
dcba	bacd

4. Construct a generating function for a_r, the number of distributions of r identical objects into:

   (a) five different boxes with at most 4 objects in each box;

   (b) four different boxes with between 3 and 8 objects in each box;

   (c) seven different boxes with at least one object in each box.

   You may leave your answers in factored form on this problem.

   Solution:

   (a) $(1 + x + x^2 + x^3 + x^4)^5$

   (b) $(x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^4$

   (c) $(x + x^2 + x^3 + \dots)^7$

5. Construct a generating function that could be used to determine how many ways there are to distribute 10 identical balls into 4 different boxes so that the first box has between 2 and 6 balls, the second box has an odd number of balls, and the other two boxes have no more than 6 balls each. The use a computer algebra system to determine the numerical answer.

   Solution: $(x^2 + x^3 + x^4 + x^5 + x^6)(x + x^3 + x^5 + x^7 + x^9)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6)^2$ We look for the coefficient of x^{10} in the expansion, which turns out to be 61, so there are 61 different distributions of balls into the boxes as described.