Solving Second Order Linear Differential Equations

MA 507

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1 The Undetermined Coefficients Method

Consider the following linear second order differential equation:

$$y'' + a(t)\, y' + b(t)\, y = f(t) \tag{1}$$

In order to describe the method of undetermined coefficients, we first need a definition.

DEFINITION. A function is called a *UC function* if it is either

1. a function of one of the following types:
   (a) t^n, where n is a non-negative integer;
   (b) $e^{\alpha t}$, where α is a non-zero constant;
   (c) $\sin(\beta t + \gamma)$, where β and γ are constants, $\beta \neq 0$;
   (d) $\cos(\beta t + \gamma)$, where β and γ are constants, $\beta \neq 0$; or
2. a function defined as a finite product of functions listed in (1).

The method of undetermined coefficients will apply when $f(x)$ in equation (1) is a finite linear combination of UC functions. Note that successive derivatives of a UC function are also UC functions or linear combinations of UC functions. This property of UC functions is what allows the method of undetermined coefficients to work.

DEFINITION. Let f be a UC function. The *UC set of* f consists of f itself together with all linear independent UC functions of which derivatives of f are constant multiples or linear combinations.

EXAMPLES

1. Let $f(t) = t^3$. Then f is a UC function, and the UC set of f is $\{t^3, t^2, t, 1\}$.
2. Let $f(t) = \sin 2t$. Then the UC set of f is $\{\sin 2t, \cos 2t\}$.
3. Let $f(t) = t^2 \cos t$. Then f is a UC function since it is a product of the functions t^2 and $\cos t$. The UC set of f is $\{t^2 \cos t, t^2 \sin t, t \cos t, t \sin t, \cos t, \sin t\}$.

Now, in the differential equation (1), suppose that $f(t)$ is a linear combination of UC functions u_i,

$$f(t) = a_1 u_1 + a_2 u_2 + \ldots + a_m u_m\,.$$

The method of undetermined coefficients prodeeds as follows.

1. Find a solution, y_u, (sometimes called a *complementary* solution) for the corresponding homogeneous (undriven) differential equation. Note that this is generally possible only if the coefficient functions $a(t)$ and $b(t)$ are constant functions. If they are not, we do not have a general procedure for finding the complementary solution.

2. For each of the UC functions
$$u_1,\, u_2,\, \ldots,\, u_m$$
of which f is a linear combination, form the respective UC sets, say
$$S_1,\, S_2,\, \ldots,\, S_m\ .$$

3. For each $i \neq j$, if $S_i \subset S_j$, delete S_i from the list.

4. For each remaining UC set which contains an element which is a solution of the homogeneous equation, multiply each member of that UC (and only that UC set) by the smallest integral power of t so that none of the members of that set solves the homogeneous equation.

5. Now form a linear combination of all the members of all the sets in step 4 using arbitrary constant coefficients (undetermined coefficients).

6. Determine the unknown coefficients by substituting the linear combination formed in the previous step into the original differential equation.

The procedure is illustrated in the following example.

Example. Solve the following differential equation using the method of undetermined coefficients.

$$y'' - 3y' + 2y = 2t^2 + e^t + 2te^t + 4e^{3t}$$

Step 1: The corresponding homogeneous (undriven) equation has solution
$$y_c = c_1 e^t + c_2 e^{2t}\ .$$

Step 2: The function $f(t)$ on the right hand side of the original differential equation is a linear combinatin of the four UC functions
$$t^2,\ e^t,\ te^t,\ \text{and } e^{3t}\ .$$
So we form the four UC sets of these functions to get $S_1 = \{t^2,\ t,\ 1\}$, $S_2 = \{e^t\}$, $S_3 = \{te^t,\ e^t\}$, and $S_4 = \{e^{3t}\}$.

Step 3: Since $S_2 \subset S_3$, we eliminate S_2 from further consideration.

Step 4: Note that S_3 includes e^t, which is a solution of the homogeneous (undriven) equation. So we multiply each member of S_3 by t to obtain $S_3' = \{t^2e^t,\ te^t\}$ which now has no members which are solutions of the homogeneous (undriven) equation.

Step 5: Forming linear combinations of the elements of S_1, S_3', and S_4, we obtain
$$y_d = At^2 + Bt + C + De^{3t} + Et^2e^t + Fte^t\ .$$

Step 6: Computing y_d' and y_d'' and substituting into the original driven ODE, one finds after some algebra that

$$y_d = t^2 + 3t + \frac{7}{2} + 2e^{3t} - t^2e^t - 3te^t \; .$$

Thus, the general solution is

$$y = y_u + y_d = c_1e^t + c_2e^{2t} + t^2 + 3t + \frac{7}{2} + 2e^{3t} - t^2e^t - 3te^t \; .$$

2 Variation of Parameters

Consider the following linear second order differential equation:

$$y'' + a(t)y' + b(t)y = f(t) \tag{2}$$

If

$$y_u(t) = c_1y_1(t) + c_2y_2(t) \tag{3}$$

is the general solution to the corresponding undriven equation, the method of *variation of parameters* allows us to search for a particular solution to Equation (2) of the form

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \tag{4}$$

The method begins by assuming that we have a general solution $y_u(t)$ to the corresponding undriven equation as in equation (3). This is no small assumption since we have only learned methods for doing this in very special cases (e.g., when $a(t)$ and $b(t)$ are constants). Nevertheless, we proceed assuming we can find a particular solution to equation (2) in the form of Equation (3). Then we can compute the first two derivatives of $y_p(t)$ and substitute them into the ODE given by equation (2). This would give us a single equation involving v_1 and v_2, so we have some latitude in imposing a second condition on those functions to simplify the work.

If we take $v_1'(t)y_1(t) + v_2'(t)y_2(t) = 0$ as the second condition, we will have two equations in the two unknown functions $v_1(t)$ and $v_2(t)$ as follows:

$$v_1'(t)y_1(t) + v_2'(t)y_2(t) = 0 \tag{5}$$
$$v_1'(t)y_1'(t) + v_2'(t)y_2'(t) = f(t) \tag{6}$$

We can use Cramer's Rule to solve the system given by equations (5) and (6) to obtain

$$v_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ f(t) & y_2'(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{\begin{vmatrix} 0 & y_2(t) \\ f(t) & y_2'(t) \end{vmatrix}}{W[y_1, y_2](t)} \tag{7}$$

$$v_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & f(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{\begin{vmatrix} y_2(t) & 0 \\ y_2'(t) & f(t) \end{vmatrix}}{W[y_1, y_2](t)} \tag{8}$$

where $W[y_1, y_2](t)$ denotes the Wronskian of $y_1(t)$ and $y_2(t)$.

EXERCISE 1: Verify this, i.e., compute the first two derivatives of y_p from equation (4), substitute them into the ODE given by equation 2 and show equation (6) follows once equation (5) is imposed.

EXERCISE 2: Use the variation of parameters method to find a particular solution of the ODE

$$2y'' - 3y' + y = 10\cos t$$

Then write the general solution of the ODE. [NOTE: This ODE is not in the form of equation (2)]

EXERCISE 3: Find a particular solution and the general solution of

$$y'' + y = \sec t$$

EXERCISE 4: Consider the ODE

$$(1-t)y'' + ty' - y = 2(1-t)^2 e^{-t}, \qquad t \in (1, \infty) \tag{9}$$

(a) Show that $\{y_1(t) = t, y_2(t) = e^t\}$ is a basic solution set for the undriven differential equation $(1-t)y'' + ty' - y = 0$ over the t-interval $(1, \infty)$.

(b) Write down the general solution of the underived differential equation given in (a) that is valid over the t-interval $(1, \infty)$.

(c) Use variation of parameters to find a particluar solution of equation (9). Then write down the general solution for this ODE.

3 Summary of Methods for Second Order DE's

Consider the second order differential equation

$$y'' + a(t)y' + b(t)y = f(t) \tag{10}$$

1. If the functions $a(t)$ and $b(t)$ are constants, it is always possible to find two linearly independent solutions of the corresponding undriven equation. A particular solution can then be found using variation of parameters. Thus, the complete solution can be found. If $f(t)$ is of the appropriate form, undetermined coefficients may be easier to use than variation of parameters to find a particular solution.

2. If the functions $a(t)$ and $b(t)$ are not constants, there is no general method for solving equation (10) in terms of a finite number of elementary functions. [Solutions involving infinite series are possible but we don't cover that topic.] However, if one solution can be found for the corresponding undriven equation, then a second solution can be found using reduction of order, and the general solution can be obtained using variation of parameters. In fact, both of these calculations can be performed simultaneously, as is illustrated below. Thus, the problem of solving $(*)$ when $a(t)$ and $b(t)$ are not constants rests on the possibility of finding one solution to the corresponding undriven equation.

EXAMPLE. Consider $ty'' + 2(1-t)y' + (t-2)y = 2e^t$. Observe that $y = e^t$ is a solution to the corresponding undriven equation. Set $y = e^t \cdot v(t)$. Then

$$[t(v'' + 2v' + v) + (2-2t)(v' + v) + (t-2)v]e^t = 2e^t$$

or

$$tv'' + 2v' = 2.$$

Then

$$v'(t) = -\frac{c_1}{t^2} + 1$$

So,

$$v(t) = \frac{c_1}{t} + t + c_2.$$

The general solution is then $y(t) = \left[\frac{c_1}{t} + t + c_2\right]e^t$.

INSTRUCTIONS. In each of the following exercises, show that the given function is a solution of the undriven ODE that corresponds with the given driven ODE. Then, find the general solution of the driven differential equation using the method of reduction of order explained in this section.

EXERCISE 5: $t^3y'' + ty' - y = 0$; $y = t$.

EXERCISE 6: $2ty'' + (1-4t)y' + (2t-1)y = e^t$; $y = e^t$.

EXERCISE 7: $(t^2+1)y'' - 2ty' + 2y = 6(t^2+1)^2$; $y = t$.