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Many different disk knots with the same exterior

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§1. Introduction

Much of codimension-two knot theory is concerned with finding and computing topological invariants of knot exteriors in order to distinguish between the knots themselves. It is well-known ([G], [L–S], [B]) that there are at most two inequivalent smooth n-sphere knots with the same exterior ($n \geq 2$), and examples of two inequivalent n-knots with the same exterior have recently been discovered ([C–S], [Go]). We show that the corresponding theory for $(n+1)$-disk knots is more complicated. Let Y denote the bounded exterior of a smooth $(n+1)$-disk knot. The *indeterminacy index* $\zeta(Y)$ is the number of inequivalent $(n+1)$-disk pairs having exteriors diffeomorphic to Y. We show that there exist disk knots with large indeterminacy indices (bigger than two, in particular). We then show that $\zeta(Y) \leq 2|\pi'|$, *where* $|\pi'|$ denotes the cardinality of π', the commutator subgroup of $\pi = \pi_1(\partial Y)$. This yields as a corollary a new and easy proof of the well-known fact that $\zeta(X) \leq 2$, where X is the exterior of an n-sphere knot, and $\zeta(X)$ its indeterminacy index.

§2. The indeterminacy index

For convenience, we work in the smooth category (the same results hold in the locally flat PL situation). We let S^n and D^{n+1} denote the standard n-sphere and $(n+1)$-disk, respectively. An *n-sphere knot* (or just *n-knot*) is the pair (S^{n+2}, kS^n) where $k: S^n \to S^{n+2}$ is an embedding. The *exterior* X of an n-knot is the complement in S^{n+2} of an open trivial 2-disk bundle neighborhood of the submanifold kS^n. An *$(n+1)$-disk knot* is the pair (D^{n+3}, gD^{n+1}) where $g: D^{n+1} \to D^{n+3}$ denotes a proper embedding, one in which the submanifold gD^{n+1} intersects ∂D^{n+3} transversely in $g(\partial D^{n+1})$. We let Y denote the $(n+1)$-disk knot exterior. Two knots are *equivalent* if there is a diffeomorphism of the ambient space throwing one submanifold onto the other (we disregard orienta-

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tions), and the *indeterminacy index* ζ is the number of inequivalent knots determined by a given knot exterior.

We will now produce examples to show that $\zeta(Y)$ can be large. The reason for this is that ∂Y contains the exterior X of the boundary sphere pair, and X can be very complicated. Recall the example of Kato [Ka 2, Theorem 4.9]:

Let $n \geq 3$, and M^{n+2} be a contractible manifold such that $\pi_1(\partial M)$ is the binary icosohedral group $G = \langle a, b \mid a^5 = b^3 = (ab)^2 \rangle$ [Ke]. Let $Y^{n+3} = S^1 \times M^{n+2}$; we will show that Y is the exterior of at least three inequivalent $(n+1)$-disk knots. Then by modifying the construction, we will show that the indeterminacy index of a disk knot exterior can be at least as large as six.

Let H be a group. A *weight element* of H is an element whose normal closure is all of H. The *automorphism class* of an element of H is the orbit of the element under the automorphism group of H. Two elements of H are *algebraically distinct* if they are in different automorphism classes.

We are interested in finding different automorphism classes of weight elements in the group $\pi_1(\partial Y) \cong Z \times G \cong \langle t, a, b \mid a^5 = b^3 = (ab)^2, ta = at, tb = bt \rangle$ where Z denotes the infinite cyclic group generated by t. An element of the form $t^n g$, for $g \in G$, is a weight element of $Z \times G$ if and only if t^n is a weight element of Z and g is a weight element of G, which forces $n = \pm 1$. To determine the weight elements of G, note that $\{1\} \lhd \{1, (ab)^2\} \lhd G$ is a composition series for G, since $\langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle$ is a presentation of the simple group A_5. The center of G is $C(G) = \{1, (ab)^2\}$, the cyclic group of order 2. Any element of G which is not in $C(G)$ is a weight element of G. The set of algebraically distinct weight elements of G is $\{a, a^2, b, b^2, ab\}$. That they are algebraically distinct follows from their different orders: 10, 5, 6, 3 and 4, respectively.

Therefore we have ta, ta^2, tb, tb^2, and tab as weight elements of $Z \times G$. However ta and ta^2 are in the same automorphism class in $Z \times G$, as are tb and tb^2 (e.g., the automorphism θ, induced by $\theta(t) = t(ab)^2$, $\theta(a) = a^7$, and $\theta(b) = ba^8 b$ sends ta to ta^2). So our list of possibly algebraically distinct weight elements is shortened to ta, tb, and tab. That these three elements are algebraically distinct follows from the fact that the center of $Z \times G$ is $Z \times \{1, (ab)^2\}$, so $Z \times G$ modulo its center is $A_5 \cong \langle a, b \mid a^5 = b^3 = (ab)^2 = 1 \rangle$. But the center is a characteristic subgroup, so any automorphism of $Z \times G$ induces one on A_5. Since a, b, and ab have different orders in A_5, their counterparts in $Z \times G$ must be algebraically distinct.

Let $\{\sigma_i \mid 1 \leq i \leq 3\}$ denote smooth embeddings of S^1 in ∂Y representing the homotopy class in ∂Y of each of the above weight elements of $Z \times G$. Choose a trivialization of the normal bundle of each σ_i, and attach 2-handles to form the manifolds $Y \cup_{\sigma_i} h^2$. The cocore or transverse disk of each 2-handle is an $(n+1)$-disk, and $(Y \cup_{\sigma_i} h^2, \text{cocore } (h^2)) \approx (D^{n+3}, g_i d^{n+1})$, where $g_i : D^{n+1} \to D^{n+3}$ is a proper smooth embedding. This is because $Y \cup_{\sigma_i} h^2$ is contractible, with simply-connected boundary, and $n+3 \geq 6$. However, no two of the three disk pairs $(Y \cup_{\sigma_i} h^2, g_i D^{n+1})$ are equivalent, because any diffeomorphism of pairs between them would restrict to a diffeomorphism on Y, inducing an isomorphism on $\pi_1(\partial Y)$ taking one of the weight elements of $Z \times G$ to another, or its inverse.

In [S], it is shown that $(n+1)$-disk pairs $(n \geq 2)$ can be constructed with an arbitrarily prescribed Alexander polynomial in a single dimension p $(2 \leq p \leq n)$, and trivial Alexander polynomials elsewhere. Moreover, these disk pairs have the property that $\pi_1(Y) \cong \pi_i(\partial Y) \cong \pi_i(S^1)$ for $i < p$. Thus, by taking the boundary connected sum of the above examples with these disk pairs, one obtains infinitely many distinct $(n+1)$-disk exteriors, each with indeterminacy index $\zeta \geq 3$. This proves

THEOREM 2.1. *For each $n \geq 3$, there exist infinitely many homeomorphically distinct $(n+1)$-disk knot exteriors Y_i, each with indeterminacy index $\zeta(Y_i) \geq 3$.*

Remark. The analogue of Theorem 2.1 for $n=2$ can be done in the topological category (non-PL embeddings). One takes $Y = S^1 \times (c * \Sigma^3)$, where $c * \Sigma^3$ is the cone on Σ^3, the Poincare′ 3-sphere. Then Y is a topological manifold [Ca], and arguments of Scharlemann [Sc] can be used to prove that the various handle attachments give rise to different non-PL disk pairs (D^5, gD^3).

We can modify the above construction to increase the lower bound for the indeterminacy index. Consider the group

$$\begin{aligned} G \times G \times G = \langle a, b, c, d, e, f \mid\ & a^5 = b^3 = (ab)^2, c^5 = d^3 = (cd)^2, e^5 = f^3 = (ef)^2, \\ & ac = ca, ad = da, bc = cb, bd = db, ae = ea, af = fa, be = eb, bf = fb, \\ & ce = ec, cf = fc, de = ed, df = fd \rangle. \end{aligned}$$

Now $G \times G \times G$ is finitely presented, and $H_1(G \times G \times G) = H_2(G \times G \times G) = 0$, so by Kervaire [Ke], for $n \geq 4$ there exists a contractible manifold M^{n+2} with $\pi_1(\partial M) \cong G \times G \times G$. As before, $Y = S^1 \times M$, and $\pi_1(\partial Y) \cong Z \times G \times G \times G$. Since the center of $Z \times G \times G \times G$ is the product of the center of each factor, we see that $Z \times G \times G \times G$ modulo its center is $A_5 \times A_5 \times A_5$. Then, as before, *tacf*, *tacef*, *tbdef*, *tace*, *tbdf*, and *tabcdef* are all algebraically distinct since their projections modulo the center have orders 15, 10, 6, 5, 3, and 2, respectively. We have the following

COROLLARY 2.2. *For each $n \geq 4$, there exist infinitely many homeomorphically distinct $(n+1)$-disk knot exteriors Y_i, each with indeterminacy index $\zeta(Y_i) \geq 6$.*

§3. An upper bound for the indeterminacy index

Now that we have seen that in some cases the lower bound of ζ can be large, we are interested in finding upper bounds. Along these lines, we have the following

THEOREM 3.1. *Let Y^{n+3} be an $(n+1)$-disk knot exterior $(n \geq 2)$. Then $\zeta(Y) \leq 2\,|\pi'|$, where $|\pi'|$ denotes the cardinality of the commutator subgroup π' of $\pi = \pi_1(\partial Y)$.*

Proof. Consider the disk pair (D^{n+3}, gD^{n+1}). Choose a trivialization $G : D^2 \times D^{n+1} \to N(gD^{n+1})$ of the tubular neighborhood of the submanifold; thus $G(\{0\} \times y) = g(y)$ for $y \in D^{n+1}$. We have that the exterior $Y = D^{n+3} - G(\mathring{D}^2 \times D^{n+1})$. Regarding $N(gD^{n+1})$ as a 2-handle attached to Y via the meridian attaching curve $G(\partial D^2 \times \{0\})$, we have $(D^{n+3}, gD^{n+1}) \approx (Y \cup_G h^2, \text{cocore}\,(h^2))$. We now wish to study the number of different ways it is possible to attach a 2-handle to Y to produce D^{n+3}. We first count the maximum number of possible isotopy classes in ∂Y of attaching curves for a 2-handle which produce a contractible manifold after handle attachment is performed. If $\pi = \pi_1(\partial Y)$, and π' is the commutator subgroup of π, we have the short exact sequence

$$1 \to \pi' \to \pi \to Z \to 1. \tag{3.2}$$

Denoting the generator of the infinite cyclic multiplicative group by t, we have a semi-direct product structure for π, and once a splitting for (3.2) is chosen, we can write each element $x \in \pi$ uniquely as $x = t^a g$ where a is an integer and $g \in \pi'$. By abuse of notation, let $t^a g$ represent an embedding of S^1 in the same homotopy class, and choose a trivialization of its normal bundle. In order for $Y \cup_{t^a g} h^2$ to be acyclic, we must have $a = \pm 1$, because $H_1(Y; Z)$ is infinite cyclic on the generator t. In order for $Y \cup_{t^a g} h^2$ to be contractible, $i_*(t^a g)$ must be a weight element of $\pi_1(Y)$, where $i_*: \pi_1(\partial Y) \to \pi_1(Y)$ is the inclusion homomorphism. In order for $\partial(Y \cup_{t^a g} h^2)$ to be simply-connected, $t^a g$ must be a weight element of $\pi_1(\partial Y)$. The upper bound we are aiming at is very crude, coming just from the homology condition $(a = \pm 1)$, so we are in fact counting the ways it is possible to complete Y to obtain an integral homology disk. The set of elements of $\pi_1(\partial Y)$ producing acyclic manifolds upon handle attachment is $\{t^{\pm 1} g \mid g \in \pi'\}$. But since the sign of the exponent of t in an element of $\pi_1(\partial Y)$ is reversed by changing the orientation of the attaching curve of h^2 (or equivalently, reversing the orientation on the cocore D^{n+1}), the set of elements of π corresponding to possibly different manifold pairs is $\{tg \mid g \in \pi'\}$, a set of the cardinality of π'. Now since we are in the dimension range $(n+2)\geq 4$ for ∂Y, homotopy of embedded one-spheres gives rise to isotopy, so the number of possible isotopy classes of attaching curves in ∂Y giving rise to acyclic manifolds is bounded above by $|\pi'|$. Now, given a represenatative of an isotopy class of attaching curves in ∂Y, there are precisely two ways to attach the 2-handle h^2, corresponding to the $\pi_1(SO)=Z_2$ ways of choosing a trivialization of the normal bundle of the curve. Hence the number of possible handle attachments yielding acyclic manifolds is bounded above by $2\,|\pi'|$.

COROLLARY 3.3. *Suppose that* Y^{n+3} $(n\geq 2)$ *is an* $(n+1)$*-disk knot exterior, and that* $\pi_1(\partial Y)=Z$. *Then* $\zeta(Y)\leq 2$, *and the two possibly different disk pairs are obtained, each from the other, by re-attaching the* 2*-handle corresponding to the normal bundle over the submanifold via the non-trivial element of* $\pi_1(SO)$.

Corollary 3.3 yields an easy proof of the well-known result that there are at most two inequivalent n-knots with the same exterior:

COROLLARY 3.4. ([B], [L–S], [Ka 1], [Sw]). *Let* $X^{n+2}\,(n\geq 3)$ *be an* n*-sphere exterior. Then* $\zeta(X)\leq 2$. *Moreover, if* $(X\cup_\gamma(D^2\times S^n),\{0\}\times S^n)$ *denotes a sphere pair obtained by sewing* $D^2\times S^n$ *onto* X *via some trivialization of the* S^n*-bundle over the meridian curve* $\gamma=S^1\times\{*\}\subset\partial X$, *then the possibly different sphere pair is* $(X\cup_{\bar\gamma}(D^2\times S^n),\{0\}\times S^n)$, *where* $\bar\gamma$ *denotes the same meridian curve with different trivialization of the* S^n*-bundle* (i.e., $D^2\times S^n$ *is sewn in with a* $\pi_1(SO)$*-twist*).

Proof. There is a one-to-one correspondence between n-sphere knots and n-disk knots with unknotted boundary $(n-1)$-sphere pair, obtained by removing an unknotted disk pair (the neighborhood of a point on the submanifold) from the sphere pair to obtain the required disk pair. An n-sphere knot and its corresponding n-disk knot have the same exterior X. But $\partial X\approx S^1\times S^n$, and $\pi_1(\partial X)=Z$, so by Corollary 2.4, $\zeta(X)\leq 2$. That is, X (thought of as a disk exterior) determines at most two inequivalent disk pairs. Therefore, thinking of it as a sphere pair exterior, then $\zeta(X)\leq 2$ as well.

§4. Some questions

1. Given a positive integer N, does there exist an $(n+1)$-disk exterior Y with $\zeta(Y)\geq N$?
2. Is there an $(n+1)$-disk exterior Y with $\zeta(Y)=+\infty$?
3. If X is an n-sphere exterior and $\pi_1(X)=Z$, must if follow that $\zeta(X)=1$?

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