Monthly Problem

L. Richard Hitt and Chad Versiga
University of South Alabama

February 14, 2007

11007. Proposed by Western Maryland College Problems Group, Westminster, MY. Let $\langle\ \rangle$ denote Eulerian numbers, and let $\{\ \}$ denote Stirling numbers of the second kind. Show that

$$\sum_{j=1}^{n} 2^{j-1} \left\langle {n \atop j} \right\rangle = \sum_{j=1}^{n} j! \left\{ {n \atop j} \right\}. \tag{1}$$

Solution: Let $\mathbb{N}_k$ denote the set of the first k positive integers. Then each term in the summation on the right-hand-side of (1) counts the number of onto functions $f : \mathbb{N}_n \twoheadrightarrow \mathbb{N}_j$. This follows because the Stirling number of the second kind, $\left\{ {n \atop j} \right\}$, counts the number of unordered partitions of a set of cardinality n into j non-empty classes. So $j! \left\{ {n \atop j} \right\}$ gives the number of such ordered partitions, which is the same as the number of onto functions $f : \mathbb{N}_n \twoheadrightarrow \mathbb{N}_j$.

In order to establish the identity, we need a connection between the number of onto functions and the Eulerian numbers. This is given in the following identity which appears in [1] using a different notation where its proof is left as an exercise.

Lemma 1.

$$j! \left\{ {n \atop j} \right\} = \sum_{k=0}^{n-1} \left\langle {n \atop k+1} \right\rangle \binom{k}{n-j}. \tag{2}$$

Proof (of lemma). Since the left-hand side of (2) counts the number of onto functions from $\mathbb{N}_n \twoheadrightarrow \mathbb{N}_j$, we show the right-hand side counts these functions as well.

So let $f : \mathbb{N}_n \twoheadrightarrow \mathbb{N}_j$ be onto. Then f can be represented as an n-tuple of integers $(a_1, a_2, \ldots, a_n)$ where $1 \le a_i \le j$ for each i and where $\{a_i | 1 \le i \le j\} = \mathbb{N}_j$ (since f is onto). Now rearrange the permutation so all the elements of $f^{-1}(1)$ occur first and in increasing order, then the elements of $f^{-1}(2)$ occur next arranged in increasing order, etc., to obtain $(a_{i_1}, a_{i_2}, \ldots, a_{i_n})$. Then $(i_1, i_2, \ldots, i_n)$ is a permutation on $\mathbb{N}_n$ with $j-1$ or fewer descents (where a descent occurs when one number in a permutation is less that its predecessor).

Now, given any n-permutation with $j-1$ or fewer descents, we use the descents as barriers to induce a partition on the n numbers. We add $j-1-k$ additional barriers, where k is the number of descents, to construct an onto function $f : \mathbb{N}_n \twoheadrightarrow \mathbb{N}_j$. This can be done by choosing where the $j-1-k$ barriers go from the $n-1-k$ available positions. This can be done in $\binom{n-1-k}{j-1-k} = \binom{n-1-k}{n-j}$ ways. Summing over all possible k gives the number of onto functions.

$$\sum_{k=0}^{n-1} \left\langle {n \atop k+1} \right\rangle \binom{n-1-k}{n-j} = \sum_{k=0}^{n-1} \left\langle {n \atop k+1} \right\rangle \binom{k}{n-j}$$

An example will serve to illustrate this counting method. A similar example is found in [2]. If we are counting the number of onto functions $f : \mathbb{N}_9 \to \mathbb{N}_6$ and consider the permutation on $\mathbb{N}_9$ given by 135274698, how many onto functions does this permutation correspond to with this counting method? If we write the permutation showing the descents, we get $135{\downarrow}27{\downarrow}469{\downarrow}8$. To define an onto function to $\mathbb{N}_6$, we must insert 2 additional barriers between adjacent numbers to create a total of 5 barriers to produce a partition into 6 ordered classes. Considering the available locations for additional barriers (denoted by $\sqcup$), $1 \sqcup 3 \sqcup 5{\downarrow}2 \sqcup 7{\downarrow}4 \sqcup 6 \sqcup 9{\downarrow}8$, we must choose 2 of the 5 without regard to order in any of $\binom{5}{2}$ ways. □

Using the lemma, we have

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$$\begin{aligned}
\sum_{j=1}^{n} j! \left\{ {n \atop j} \right\} &= \sum_{j=1}^{n} \sum_{k=0}^{n-1} \left\langle {n \atop k+1} \right\rangle \binom{k}{n-j} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle \binom{k-1}{n-j} \\
&= \sum_{k=1}^{n} \sum_{j=1}^{n} \left\langle {n \atop k} \right\rangle \binom{k-1}{n-j} \\
&= \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle \sum_{j=1}^{n} \binom{k-1}{n-j} \\
&= \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle 2^{k-1}.
\end{aligned}$$

□

References

[1] R.L. Graham, D.E. Knuth, and O. Patashnik, Concrete Mathematics, Second Edition, Addison-Wesley, Reading, MA, 1994.

[2] Donald E. Knuth, The Art of Computer Programming, Volume 3: Sorting and Searching, Second Edition, 1998, Addison-Wesley.