Wallis' Rule

As you have probably observed in class and in your homework exercises, you are often confronted with integrating a power of sine or cosine over the interval from 0 to $\frac{\pi}{2}$. Later in the chapter on multiple integration, we will also need to integrate powers of sine times powers of cosine from 0 to $\frac{\pi}{2}$. This handout provides a formula that evaluates such integrals immediately.

In Section 5.6 of the text on integration by parts, techniques for evaluating integrals of the form $\int \sin^n t\,dt$ and $\int \cos^n t\,dt$ are discussed. [See Example 6 on page 406 and Exercises 33–36 on page 407.] Example 6 there derives a so-called reduction formula for integrating $\sin^n x$ for $n \geq 2$:

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

The power of the formula is that it *reduces* the exponent on the sine function that you have to integrate – thus the name *reduction formula*. A similar reduction formula is in Exercise 33 on page 407. Exercise 36 (page 407) uses the reduction formula from Example 6 to derive formulas for definite integrals of even powers of sine on the interval from 0 to $\frac{\pi}{2}$. These formulas for the definite integrals of powers of sine can be generalized at least to the following extent:

$$\int_0^{\frac{\pi}{2}} \sin^m t \cos^n t\,dt = \frac{(m-1)\cdot(m-3)\cdot(m-5)\cdots(2 \text{ or } 1)\cdot(n-1)\cdot(n-3)\cdot(n-5)\cdots(2 \text{ or } 1)}{(m+n)\cdot(m+n-2)\cdot(m+n-4)\cdots(2 \text{ or } 1)}\cdot\alpha$$

$$\text{where } \alpha = \left\{\begin{array}{ll} \frac{\pi}{2} & \text{if } m \text{ and } n \text{ are both even} \\ 1 & \text{otherwise} \end{array}\right\}$$

This formula is attributed to the English mathematician John Wallis (1616 – 1703). Its proof is also based on reduction formulas.

To practice using the formula, show each of the following:

1. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4 t \cos^6 t\,dt = \frac{45\pi}{7680}$
2. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 t \cos^4 t\,dt = \frac{2}{35}$
3. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 t \cos^{10} t\,dt = 0$
4. $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6 t\,dt = \frac{15\pi}{96}$
5. $\displaystyle\int_0^{\pi} \sin^3 t \cos^3 t\,dt = 0$
6. $\displaystyle\int_0^{\pi} \sin^3 t \cos^4 t\,dt = \frac{4}{35}$