MA 367-01	**Quiz #1 Solutions**	**24 January 2003**

INSTRUCTIONS: Turn in solutions to the following problems by Tuesday (28 January 2003) at Noon. You may give your solutions directly to me, put them under my office door, or place them in my mailbox. E-mail is also acceptable. Fully explain your solutions and calculate the numerical values.

1. Fifteen people are arranged in an order.

   (a) In how many ways can this be done?

   (b) If individuals of the same gender must be adjacent, and if there are 10 women and 5 men, in how many ways can this be done?

   (c) In how many ways can this be done if Jim must be placed (somewhere) before Liam who in turn must come (somewhere) before Lawanda?

   (d) In how many ways can the 15 be arranged so that Jim and Lawanda are not adjacent?

   Solution:

   (a) Fifteen people can be arranged in a row in $P(15,15) = 15! = 1,307,674,368,000 \approx 1.307674368 \times 10^{12}$ ways.

   (b) Any such arrangement can be accomplished by performing three consecutive events: arrange the genders, arrange the men, then arrange the women. So this can be done in $2 \cdot 10! \cdot 5! = 870,912,000$ ways.

   (c) First choose the three locations for the designated individuals (C(15,3) ways). Then place the three designated individuals in their positions (1 way). Then place the remaining twelve people in the twelve remaining positions (12! ways). So such an arrangement can be done in $C(15,3) \cdot 12! = 217,945,728,000 \approx 2.17946 \cdot 10^{11}$ ways.

   (d) Since the number of ways two designated individuals are adjacent is $2 \cdot 14!$, the number of ways they are not adjacent is $15! - 2 \cdot 14! = 1,133,317,785,600 \approx 1.13332 \times 10^{12}$.

2. From a group of 5 men and 5 women,

   (a) in how many (unordered) ways can they be paired up to form five couples (each couple contains one person from each gender)?

   (b) in how many (unordered) ways can they be paired up to form five pairs (without regard to gender)?

   Solution:

   (a) Each man needs to be assigned to one woman without repetition (or vice versa, if you prefer). There are five ways to assign the first man, four for the second, etc. So there are $5! = 120$ ways to pair up the couples. This can also be thought of as the number of one-to-one functions from the set of men to the set of women.

   (b) *Solutions 1.* For a simple-minded approach, we could just line up the 10 people in a row and pair up the first two, then the second two, etc. Of course, this method gives rise to a horrible overcount since many of the 10! ways of lining up 10 people produce equivalent parings. Since we can interchange the first 2 in line and not get a distinguishable pairing, and the same for the second 2, etc., we need to divide the 10! count by 2^5. But since we can also permute the 5 pairs in any way and still not get a distinguishable pairing, we also need to divide by 5!. This gives $\frac{10!}{5! \times 2^5} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{2^5 5!} = \frac{5 \times 9 \times 4 \times 7 \times 3 \times 5 \times 2 \times 3 \times 1}{5!} = 9 \times 7 \times 5 \times 3 = 945$.

Solutions 2. Because of the nice form of the answer ($9 \times 7 \times 5 \times 3$), you might expect there is a more direct counting method. For example, you could take any one of the 10 people. Then there are 9 other people with whom that person could be paired. For a person other than the two already paired, there are 7 with whom that person can be paired. And so on. This gives rise directly to the $9 \times 7 \times 5 \times 3$ result.

Solutions 3. For a final, and perhaps easiest, solution: choose a pair from the 10 in $C(10, 2)$ ways. Now choose a second pair from the remaining 8 in $C(8, 2)$, and so on. But since the order of the pairs is not supposed to be counted, we have to divide this product by the number of ways the pair could be ordered (5!), which gives $\frac{C(10,2)\times C(8,2)\times C(6,2)\times C(4,2)\times C(2,2)}{5!}$ which gives the same answer as before.

3. A student must answer 5 out of 10 questions on a test. The five questions that are answered must include at least 2 of the first 5 questions. If the order of the answers is not important, in how many different ways can this be done?

Solution: Be careful. If you choose 2 questions of the first 5 (C(5,2)) and then choose 3 of the remaining 8 (C(8,3)), you have counted many selections multiple times.

Solution 1. A direct count has to be broken into several cases. Think of having the five questions that are answered written in numerical order.

Case 1. If the second question answered is 2 (i.e., questions 1 and 2 are answered): there are $C(8, 3)$ ways to select the other three questions.

Case 2. If the second question answered is 3 (i.e., either 1 and 3 are answered or 2 and 3 are answered): there are $C(7, 3)$ ways to select the other three questions.

Case 3. If the second question answered is 4 (3 ways): there are $C(6, 3)$ ways to select the other three questions.

Case 4. If the second question answered is 5 (4 ways): there are $C(5, 3)$ ways to select the other three questions.

So the total number is $C(8, 3) + 2 \times C(7, 3) + 3 \times C(6, 3) + 4 \times C(5, 3) = 226$

Solutions 2. This problem can also be solved, perhaps more easily, by counting the complementary event and subtracting that number from the total. The total is just $C(10, 5)$, while the number of ways of selecting zero or one of the first five questions and the rest from the second five is $C(5, 0) \times C(5, 5) + C(5, 1) \times C(5, 4) = 1 + 5 \times 5 = 26$, so there are $C(10, 5) - 26 = 252 - 26 = 226$ ways of selecting 2 or more of the first five questions.

4. How many different 5-card hands consist of
   (a) three cards of one kind and two cards of another (a full house)?
   (b) exactly two pairs (i.e., a pair of cards from each of two different kinds with no other match in kind)?

Solution:

(a)
   i. Select 2 kinds with regard to order ($P(13, 2) = 13 \times 12$);
   ii. Select 3 cards from the first kind (C(4,3)=4);
   iii. Select 2 cards from the second kind (C(4,2)=6);

   So the total number of full houses is $13 \times 12 \times 4 \times 6 = 3,744$.

(b)
   i. Select two kinds from thirteen without regard to order (you can't tell the difference in the two kinds because they will both have two cards) (C(13,2)=78);
   ii. Select two cards from each of the selected kinds ($C(4, 2)^2 = 6^2 = 36$).
   iii. Select a fifth cards from the remaining 11 kinds (44).

   So there are $78 \times 36 \times 44 = 123,552$ different hands that contain exactly two pairs.