MA 367-01 §5.3 – 5.4	# Quiz #2 Solutions	31 January 2003

INSTRUCTIONS: Turn in solutions to the following problems by Wednesday (5 February 2003) in class. Fully explain your solutions and calculate the numerical values.

1. In how many ways can 15 identical pieces of candy be distributed to 4 children

   (a) with no restrictions?

   (b) so that each child gets at least 2 pieces of candy?

   (c) in addition, so that no child gets more than 8?

   Solution:

   (a) We can distribute 15 identical balls into 4 distinguishable boxes in $C(15+4-1, 15) = 816$ ways.

   (b) We first put two of the balls into each of the boxes, then distribute the remaining 7 balls in $C(7+4-1, 7) = 120$ ways.

   (c) If each child gets at least 2 and no child gets more than 8, we have to take the 120 solutions to the previous part and delete all those in which a child get 9 or more pieces of candy. But there are only 4 such, since $9+2+2+2 = 16$. Thus there are $120-4 = 116$ distributions.

2. Find the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 2$ where $x_i \geq -5$ for each i.

   Solution: This is equivalent to counting the number of solutions to the equation $y_1+y_2+y_3+y_4 = 22$ where each $y_i \geq 0$, which gives $C(22+4-1, 22) = 2,300$.

3. A downtown area consists of a large square area that is 10 blocks by 10 blocks in size. If a car starts at the southwest corner and ends at the northeast corner traveling only east and north, how may different routes are possible that involve 4 or fewer turns?

   Solution: The car must travel 10 blocks east and 10 blocks north, so we can model this with interleaved sequences of 10 E's and 10 N's (or 0's and 1's, if you like) that have no more than 4 changes. This count naturally breaks down into cases.

   1 Turn There are only 2 paths that have one turn.

   2 Turns The sequence can be of the form ENE or NEN. We can count each as the number of ways to distribute 10 balls in 2 boxes so each box gets at least one ball, $C(8+2-1, 8) = 9$. This gives a total of 18 for both configurations.

   3 Turns Count the configurations ENEN (then double that count to get the NENE in addition). We have to distribute the 10 E's in two boxes so no box is empty, and likewise with the N's, so we get $C(10-2+2-1, 10-2) \times C(10-2+2-1, 10-2) = 9 \times 9 = 81$ paths of type ENEN. Doubling this gives 162.

   4 Turns Count the ENENE configurations and double the result to include the NENEN's. This time the E's go into 3 boxes and the N's into 2, so we get $C(10-3+3-1, 7) \times C(10-2+2-1, 8) = 36 \times 9 = 324$. Similarly for the NENEN, so we get a total of 648.

   Adding up the cases, we get $2+18+162+648 = 830$ different paths with 4 or fewer turns.

4. Let D and R be sets with $|D| = k$ and $|K| = n$.

   (a) How many functions are there from D to R ($f : D \to R$)?

(b) How many such functions are injective (one-to-one)? Be sure to address all cases for n and k.

(c) How many such functions are surjective (onto)? OK, this problem is too difficult right now, so let's settle for counting this in the special case that $k = 5$ and $n = 2$. The let $n = 3$ and count them again.

Solution:

(a) For each of the k elements in D, choose one of the n elements in R. So there are n^r functions.

(b) This time the number of choices is reduced after each assignment, so we get $P(n, r)$ ways of making an ordered selection of r of n distinct objects if $n \geq r$. If $n < r$ we get 0 since no function $f : D \to R$ is 1-1.

(c) There are 2^5 functions from a set with 5 elements to a set with 2 elements. All but 2 of these are onto since there are only 2 ways to send the 5 domain elements to just 1 of the range elements. This gives $32 - 2 = 30$ onto functions. You can also count this directly by looking at the number of ways to partition the 5 domain elements into two non-empty ordered classes: $C(4, 1) + C(3, 2) + C(2, 3) + C(1, 4) = 30$.

If we count onto functions from a 5-element set to a 3-element set, we can distinguish 2 cases by looking at the ways the domain is partitioned by the onto function:

(3,1,1), (1,3,1), (1,1,3) The domain set is partitioned into 3 ordered non-empty classes of sizes 3, 1, and 1. Each of these configurations occurs in $P(5; 3, 1, 1)$ ways, so we get $3 \times 20 = 60$ such onto functions.

(2,2,1),(2,1,2),(1,2,2) Similarly, we get $3 \times P(5; 2, 2, 1) = 3 \times 30 = 90$ such onto functions.

So the total number of onto functions is $60 + 90 = 150$.

5. How many ways can 10 identical pieces of candy be placed in 3 identical bags? One way to do this is to enumerate all the possibilities in some systematic manner. There is no known closed-form formula for the general problem of distributing identical balls into identical boxes, although we can make progress on a recurrence relation and on a generating function that "solves" this kind of problems.

Solution: This amounts to the number of unordered ways you can add 3 non-negative integers to get 10. Its easy to just enumerate them: 10+0+0, 9+1+0, 8+2+0, 8+1+1, 7+3+0, 7+2+1, 7+3+0, 6+4+0, 6+3+1, 6+2+2, 5+5, 5+4+1, 5+3+2, 4+4+2. Total: 14.