MA 367-01 §5.1–6.3	**Test #1 Solutions**	**Name:** ______________ **24 February 2003**

INSTRUCTIONS: You do not need to compute numerical values for your answers, but you must express them using using arithmetic operations and the factorial operator. Explain how you obtain your answers unless otherwise directed.

1. *(25 points)* SHORT ANSWER: You just need to write the answer on this one – no explanation is necessary.

   (a) If $|A| = n$, the total number of subsets of A is $\underline{\quad 2^n \quad}$.

   (b) If $|A| = n$, the number of subsets of A of size k is $\underline{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$.

   (c) The number of distinguishable ways can 3 A's, 4 B's, and 6 C's be all arranged in a row is $\underline{\quad P(13; 3, 4, 6) = \frac{13!}{3!4!6!} = 60060 \quad}$.

   (d) The number of ways of selecting k objects without repetition from n distinguishable objects ($k \le n$) and arranging them in a row is $\underline{P(n, k) = \frac{n!}{(n-k)!}}$.

   (e) The infinite series $1 + x + x^2 + x^3 + \cdots$ can be formally written in closed form as $\underline{\quad \frac{1}{1-x} \quad}$.

2. *(15 points)* How many ways can a group of 5 girls and 4 boys be lined up in a row

   (a) (with no restriction)?

   (b) if the boys all have to be adjacent and the girls have to be adjacent?

   (c) if no two girls can be adjacent?

   Solution:

   (a) $9! = 362880$

   (b) Group the boys together and arrange them in 4! ways; group the girls together and arrange them in 5! ways; then choose an order on the two groups in 2 ways. Performing the events successively yields $4!5!2 = 5760$.

   (c) Since there must be a boy between each girl, we arrange the girls in 5! ways, then the boys in 4! ways, to get $5!4! = 2880$. Had the number of girls been more than one larger than the number of boys, this could have been a nasty problem.

3. *(14 points)* A coin is tossed 12 times.

   (a) How many ways can this result in 9 heads and 3 tails?

   (b) Of those, how many contain at least 5 heads in a row?

   Solution:

   (a) Select which of the 12 tosses results in heads for a total of $\binom{12}{9} = \frac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 220$.

   (b) This is like Exercise #17 from §5.4. Consider the three T's separated by boxes (possibly empty) of H's: $\sqcup T \sqcup T \sqcup T \sqcup$. Put 5 H's in one of the four boxes (4 ways to do this). Then distribute the remaining 4 H's in the four boxes in $\binom{4+4-1}{4} = 35$ ways. So there are $4 \times 35 = 140$ ways to get 5 or more heads in a row if there are exactly 9 heads in 12 tosses.

4. *(11 points)* In how many ways can 10 identical balls be distributed into 4 distinguishable boxes so that the first box has an odd number of balls, the second box has an even number of balls, and the last two boxes have between 2 and 5 balls (inclusive)? Don't actually compute this number, but describe how it could be computed using generating functions.

   Solution: Let $f(x) = (x + x^3 + x^5 + x^7 + x^9)(1 + x^2 + x^4 + x^6 + x^8 + x^{10})(x^2 + x^3 + x^4 + x^5)^2$ and determine the coefficient of the x^{10} term in the expansion.

5. *(15 points)* How many ways can 12 jobs be assigned to 3 different people if the jobs are indistinguishable for counting purposes

   (a) ?

   (b) if each person gets at least one one job?

   (c) if each person gets at least three jobs?

   Solution:

   (a) Distribute 12 identical balls into 3 distinguishable boxes in $\binom{12+3-1}{12} = 91$ ways.

   (b) After putting one ball in each box, distribute the remaining 9 balls in $\binom{9+3-1}{9} = 55$ ways.

   (c) After putting 3 balls in each box, distribute the remaining 3 in $\binom{3+3-1}{3} = 10$ ways.

6. *(10 points)* Use the binomial theorem to show that the number of subsets of a set A having an odd number of elements is the same as the number of subsets of A having an even number of elements.

   Solution: Beginning with the binomial theorem, $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, substitute $x = -1$ to obtain

$$(1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}.$$

   Note that the left-hand-side equals 0. Interpreting the binomial coefficient $\binom{n}{k}$ as the number of subsets of A having cardinality k, the result follows.

7. *(10 points)* Let $p(m,n)$ denote the number of integer partitions of m into exactly n parts. Complete the following table of values for $p(m,n)$ using the property that a value in a row can be expressed as a sum of values in a particular previous row. Explain in detail how $p(8,3)$ is computed as a specific example.

m / n	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
1	1	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0
3	1	**1**	1	0	0	0	0	0
4	1	**2**	**1**	1	0	0	0	0
5	1	**2**	**2**	**1**	1	0	0	0
6	1	**3**	**3**	**2**	**1**	1	0	0
7	1	**3**	**4**	**3**	**2**	**1**	1	0
8	1	**4**	**5**	**5**	**3**	**2**	**1**	1

$p(8,3) = p(5,1) + p(5,2) + p(5,3) = 1 + [p(3,1) + p(3,2)] + [p(2,1) + p(2,2)] = 1 + [1 + p(1,1)] + [1 + 1] = 5.$