MA 367-01 §7.1–8.3	**Test #2 Solutions**	**Name:** ________________ **23 April 2003**

INSTRUCTIONS: Put all answers/solutions on a separate sheet. Explain how you obtain your answers unless otherwise directed.

1. *(36 points)* SHORT ANSWER: Answer each of the following and explain what you did.

   (a) In how many ways can 5 pairs of parentheses be arranged in a row that are algebraically correct?

   Solution: This is the Catalan number $\frac{1}{6}\binom{10}{5} = 42$

   (b) Write the rook polynomial for the board $\begin{array}{|c|c|}\hline X & X \\ \hline X & X \\ \hline\end{array}$ (the X denotes shading).

   Solution: $R(x) = 1 + 4x + 2x^2$ since there are 4 ways to place one rook on the board and just two ways to place two rooks on the board.

   (c) Compute a Stirling Number of the Second Kind $S(6,3)$, and give a combinatorial explanation of what this number counts.

   Solution: $S(6,3) = 90$ (computed, e.g., from recurrence relation) counts the number of set partitions of a set with 6 elements into 3 unordered non-empty subsets.

   (d) Compute the derangement number, D_{10}, the number of derangements of 10 objects.

   Solution: Use one of the recurrence relations to compute the derangement sequence beginning with, say, $n = 1$: 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961 (where the recurrence relation $D_n = (n-1)(D_{n-1} + D_{n-2})$ has been used). Alternatively, the formula $D_n = \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^n = 1334961$ could be used.

   (e) Find the number of onto functions $f : \mathbb{N}_5 \to \mathbb{N}_3$.

   Solution: The number of onto functions in this case is $3! * S(5,3) = 6 * 25 = 150$.

   (f) If the rook polynomial for the shaded (forbidden) positions in a 5×5 chessboard is $1 + 13x + 56x^2 + 96x^3 + 60x^4 + 12x^5$, in how many ways can 5 mutually non-attacking rooks be placed on the unshaded squares?

   Solution: Using the coefficients from the rook polynomial to work the corresponding inclusion-exclusion problem, we get $5! - 13 \times 4! + 56 \times 3! - 96 \times 2! + 60 \times 1! - 12 \times 0! = 0$, so there aren't any ways to arrange the rooks on the board. This polynomial came from multiplying the rook polynomial for a fully shaded 2×2 board by the rook polynomial for a fully shaded 3×3 board, which explains (when you think about it, perhaps) why there aren't any ways to arrange these rooks.

2. *(16 points)* Let a_n denote the number of n-digit ternary (digits are 0, 1, or 2) sequences that do not have two consecutive zeros. Determine (and fully explain) a recurrence relation with initial conditions for this sequence. Do not solve this recurrence relation.

   Solution: you can partition all such n-digit sequences into two disjoint sets.

   Case 1 The n-digit number begins with a 0. In this case you need to make sure the next digit is not zero (2 choices) followed by any ($n-2$-digit sequence not containing consecutive zeros (a_{n-2} choices).

   Case 2 the n-digit number does not begin with 0 (hence begins with 1 or 2 – two choices). Then you can follow this first digit with any of the $(n-1)$ digit sequences that do not have consecutive zeros (a_{n-1} ways).

   This gives the recurrence relation $a_n = 2\,a_{n-1} + 2\,a_{n-2}$ with $a_1 = 3$ and $a_2 = 8$.

3. *(16 points)* Find the general solution for the recurrence relation $a_n = 3\,a_{n-2} + 2\,a_{n-1}$. Then find the solution that satisfies the initial conditions $a_0 = 3$ and $a_1 = 1$.

Solution: The characteristic equation is $r^2 - 2r - 3 = 0$ with roots $r = -1$ and $r = 3$, so the general solution is $a_n = A\,3^n + B(-1)^n$. Applying the initial conditions gives $A = 1$ and $B = 2$, so the solution $a_n = 3^n + 2\,(-1)^n$.

4. *(16 points)* Use inclusion-exclusion to find the number of ways to distribute 20 identical pieces of candy among four combinatorics students if no student can get no more than 5 pieces of candy. Show the details of your inclusion-exclusion counting method.

   Solution: There are $\binom{23}{20}$ ways to distribute the candy without restriction. If we let A_i denote the collection of distributions in which student i gets more than 5 pieces of candy, we have $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| = |A_1 \cup A_2 \bar{\cup} A_3 \cup A_4| = |U| - A_1 \cup A_2 \cup A_3 \cup A_4| = \binom{23}{20} - |A_1 \cup A_2 \cup A_3 \cup A_4| = \binom{23}{20} - \binom{4}{1}\binom{17}{14} + \binom{4}{2}\binom{11}{8} - \binom{4}{3}\binom{5}{2} + 0 = 1771 - 4 \times 680 + 6 \times 165 - 4 \times 10 = 1$ which establishes the rather obvious fact that there is just 1 way to give 20 pieces of candy to 4 combinatorial students so that no student gets more than 5 pieces.

5. *(16 points)* Find the rook polynomial for the following forbidden position problem. You may leave the polynomial in factored form, and you need not go any farther with the problem than finding the rook polynomial.

   We want to find the number of ways 5 people (A, B, C, D and E) can be assigned 5 tasks (1, 2, 3, 4 and 5) to do if person A cannot do tasks 1 and 2, person B cannot do tasks 2 and 4, person C cannot do tasks 1 and 2, and person D cannot do tasks 3 and 4, and person E cannot do tasks 4 and 5.

   Solution:

Rewrite the board

	1	2	3	4	5
A	X	X			
B		X		X	
C	X	X			
D			X	X	
E				X	X

as

	1	2	3	4	5
A	X	X			
C	X	X			
B		X		X	
D			X	X	
E				X	X

Now apply the reduction formula at entry B2 (B4 also works well) to obtain

$$R(C) = R(C1_s) + x\,R(C2_s^*) = R\left(\begin{array}{|c|c|c|c|c|}\hline X & X & & & \\ \hline X & X & & & \\ \hline & & & X & \\ \hline & & X & X & \\ \hline & & & X & X \\ \hline\end{array}\right) + x \cdot R\left(\begin{array}{|c|c|c|c|c|}\hline X & & & & \\ \hline X & & & & \\ \hline & & & & \\ \hline & & X & X & \\ \hline & & & X & X \\ \hline\end{array}\right)$$

$$= R\left(\begin{array}{|c|c|}\hline X & X \\ \hline X & X \\ \hline\end{array}\right) \cdot R\left(\begin{array}{|c|c|c|}\hline & X & \\ \hline X & X & \\ \hline & X & X \\ \hline\end{array}\right) + x \cdot R\left(\begin{array}{|c|}\hline X \\ \hline X \\ \hline\end{array}\right) \cdot R\left(\begin{array}{|c|c|c|}\hline X & X & \\ \hline & X & X \\ \hline\end{array}\right)$$

$$= (1 + 4x + 2x^2)(1 + 5x + 5x^2 + x^3) + x(1 + 2x)(1 + 4x + 3x^2)$$

$$= 1 + 10x + 33x^2 + 42x^3 + 20x^4 + 2x^5$$

You didn't have to multiply out the last step on the test. And just for curiosity, the number of ways to place 5 non-attacking rooks on this board is

$$5! - 10 \times 4! + 33 \times 3! - 42 \times 2! + 20 \times 1! - 2 \times 0! = 12$$