There Exist Arbitrarily Many Different Disk Knots with the Same Exterior
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THERE EXIST ARBITRARILY MANY DIFFERENT DISK KNOTS WITH THE SAME EXTERIOR

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ABSTRACT. We prove that, for $n \geq 5$, exteriors of disk knots of $D^n$ in $D^{n+2}$ can be exteriors of arbitrarily many different disk knots.

1. Introduction. In [H-S], we showed that there are at least three different disk knots of $D^4$ in $D^6$ with the same exterior, and at least six different disk knots of $D^n$ in $D^{n+2}$ with the same exterior for $n \geq 5$. We improve the latter result here by showing that there exist arbitrarily large classes of inequivalent disk knots with the same exterior.

Let $Y$ denote the bounded exterior of a smooth $n$-disk knot. The indeterminacy index $\zeta(Y)$ is the number of inequivalent $n$-disk pairs having exteriors diffeomorphic to $Y$. We prove the following

THEOREM. Let $n \geq 5$. Given a positive integer $N$, there exists an $n$-disk knot exterior $Y$ with $\zeta(Y) \geq N$.

This answers Question 1 in [H-S] in the affirmative.

2. The construction. For convenience, we work in the smooth category, although the results hold in the locally flat PL category as well. An $n$-disk knot is a manifold pair $(D^{n+2}, f(D^n))$ where $f : D^n \to D^{n+2}$ denotes a proper embedding in which the submanifold $f(D^n)$ intersects $\partial D^{n+2}$ transversely. The exterior $Y$ of an $n$-disk knot is the complement in $D^{n+2}$ of a trivial open 2-disk bundle neighborhood of the submanifold $f(D^n)$. Two disk knots are equivalent if they are diffeomorphic as unoriented pairs, i.e., if there is a diffeomorphism of $D^{n+2}$ onto itself which sends one submanifold onto the other (disregarding orientations).

The construction used in [H-S] to show that $\zeta(Y)$ can be as large as six was a modification of an example of Kato [Ka, Theorem 4.9]. We recall the construction and further modify it as follows.

Let $G$ be any finitely presented group with $H_1(G; \mathbb{Z}) = H_2(G; \mathbb{Z}) = 0$. If $n + 1 \geq 6$, then Kervaire [Ke] has shown that there is a contractible manifold $M^{n+1}$ with $\pi_1(\partial M) = G$. Suppose also that $G$ has weight one (i.e., $G$ has an element, called a weight element, whose normal closure is all of $G$). Then form the manifold $Y = S^1 \times M^{n+1}$. We see that $Y$ is a disk knot exterior by attaching a 2-handle $h^2$ to $Y$ along the path $tg \in \pi_1(\partial Y) = J \times G$, where $g$ is a weight element of $G$, and $t$ is a generator of the infinite cyclic factor $J$. In this case, $tg$ is a weight...
element of $\partial Y)$, and $(Y \cup h^2, \text{cocore}(h^2))$ is an $n$-disk knot. If now $J \times G$ has many different weight elements, this gives rise to different handle attachments, and possibly inequivalent $n$-disk knots.

To help measure the inequivalency, we call two elements $a, b$ in a group $H$ \textit{algebraically distinct} if the orbit of the set $\{a, a^{-1}\}$ under all automorphisms of $H$ is disjoint from the orbit of the set $\{b, b^{-1}\}$. Then any two algebraically distinct weight elements of $J \times G$ give rise to inequivalent disk knots in this construction. Thus, the proof of the theorem is reduced to finding a suitable class of groups to use in the role of the group $G$.

3. The special linear groups. In [H-S], we used the group $G = \langle a, b^5 = b^3 = (ab)^2 \rangle = SL(2, 5)$ to obtain a group $J \times G$ with three algebraically distinct weight elements, and observed that $J \times G \times G \times G$ contains at least six algebraically distinct weight elements. Here, we use $SL(2, p)$ for $p$ a prime, $p \geq 5$.

Recall that $SL(2, p)$ is its own commutator subgroup (see, e.g., [D, pp. 38–40]), so $H_1(SL(2, p)) = 0$. And, as Gordon [G] points out, $H_2(SL(2, p)) = 0$ [S, p. 95, Corollary 2]. Furthermore, $Z(SL(2, p)) = \{1, -1\}$ where $Z(G)$ denotes the center of the group $G$ and $I$ denotes the identity matrix; and, any noncentral element of $SL(2, p)$ is a weight element (e.g. [R, p. 159]). Thus, any element of the form $tg \in J \times SL(2, p)$, where $t$ generates $J$ and $g$ is not in the center of $SL(2, p)$, is a weight element of $J \times SL(2, p)$. Now let $[a]$ denote the matrix

$$
\begin{bmatrix}
a & 0 \\
0 & a^{-1}
\end{bmatrix} \in SL(2, p)
$$

and let $[a]$ denote the equivalence class of $[a]$ in the group $PSL(2, p)$. Since any automorphism of $J \times SL(2, p)$ induces one on

$$
\frac{J \times SL(2, p)}{Z(J \times SL(2, p))} \cong \frac{SL(2, p)}{Z(SL(2, p))} \cong PSL(2, p),
$$

we can show that there are algebraically distinct weight elements in $J \times SL(2, p)$ by showing that their projections in $PSL(2, p)$ are algebraically distinct. But the order of $[a] \in PSL(2, p)$ is the same as the order of $a$ in the multiplicative group $\mathbb{Z}_p^*$ of the field with $p$ elements $\mathbb{Z}_p$; and, the order of $[a]$ is the same as the order of $[a]^{-1}$. Moreover, since $\mathbb{Z}_p^*$ is cyclic, given any divisor of its order $p - 1$, there is an element in $\mathbb{Z}_p^*$ of that order. The Theorem then follows once it is noted that $\lim \sup \{\tau(p - 1) | p \text{ prime}\} = +\infty$ where $\tau(p - 1)$ denotes the number of divisors of $p - 1$. But this follows from Dirichlet’s Theorem, which implies that for any integer $k$, there is a prime of the form $1 + km$.

F. González-Acuña [G-A] has pointed out that it follows from Huppert [H, Seite 646, Satz 25.7] that $H_2(SL(2, 2^p - 1)) = 0$ for $p$ prime, $p \geq 5$. Also, $H_1(SL(2, 2^p - 1)) = 0$ since $SL(2, 2^p - 1)$ is simple. Dirichlet’s Theorem can also be used here to show that $\lim \sup \{\tau(2^p - 1) | p \text{ prime}\} = +\infty$.

Thus, either of the classes of groups $SL(2, p)$, $SL(2, 2^p - 1)$ for $p$ prime, $p \geq 5$, can be used in the role of $G$ in the construction. This completes the proof of the theorem.

As in [H-S], any of the above $n$-disk knot exteriors can be modified by taking the boundary connected sum with an $n$-disk knot having arbitrarily prescribed Alexander polynomial in a single dimension $k$ ($2 \leq k \leq n - 1$) and trivial Alexander polynomial.
polynomial elsewhere [Su]. This produces an infinite class of \( n \)-disk knot exteriors, each having indeterminacy index at least that of the original \( n \)-disk knot exterior. Thus we have the following

**COROLLARY.** Let \( n \geq 5 \). Given a positive integer \( N \), there exist infinitely many homeomorphically distinct \( n \)-disk knot exteriors, each having indeterminacy index greater than \( N \).

**NOTE.** We have recently learned that F. Gonzalez-Acuña and S. Plotnick (independently) have produced examples with \( \gamma(Y) = +\infty \) (private communications).

**REFERENCES**


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