THE FLORIDA STATE UNIVERSITY

COLLEGE OF ARTS AND SCIENCES

HANDLEBODY PRESENTATIONS OF KNOT COBORDISMS

BY

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A Dissertation submitted to the
Department of Mathematics
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

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December, 1977

ACKNOWLEDGMENTS

There are many people who have contributed to my graduate education. They include, but are not limited, to the following.

A debt of gratitude is due my parents, who have always stood behind me, and encouraged me in my studies.

Thanks is also due the National Science Foundation, and the Florida State University Mathematics Department for making my graduate education possible.

I would also like to thank Robin Lucarelli for the excellent typing of the dissertation, and Dr. Susan Conlon for assisting with the typing.

I am very grateful for Dr. Robert Kromhout and Dr. Nicholas Heerema serving on my committee. To Dr. John Bryant, who served on the committee, and who contributed to my interest in topology, I am also very appreciative. I would like to express my appreciation to Dr. Chris Lacher for serving on my committee, and for serving as my advisor during a difficult year for me. Most of all, I would like to thank Dr. DeWitt Sumners, my major professor, for his friendship and guidance. His energy and love for mathematics have been very inspiring.

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INTRODUCTION AND PRELIMINARIES

Introduction

Knot cobordism is an equivalence relation on the set of n-knots under which the set of equivalence classes forms an abelian group using the connected sum operation. The general idea in studying handle presentations of knot cobordisms is to see which geometric conclusions can be made concerning two cobordant knots, if assumptions are made about the handle presentation of a cobordism between them. Conversely, geometric assumptions can be made about a pair of cobordant knots, and the implications of these assumptions concerning the handle structure of a corbordism between the knots can be investigated.

Handle theory has been studied extensively over the past two decades. A handle presentation is a compact way of presenting geometric information in the same way that a group presentation conveys algebraic information. Moreover, the cores of the handles are lower in dimension than the surrounding space, and are therefore easier to deal with. The main idea in this work is that information about knots can be obtained by examining handle presentations of cobordisms which they bound.

In Chapter I, classical ribbon knots are discussed, and the concept of ribbon knots is generalized to higher dimensions following Yanagawa [40, 41, 42] and Roseman [28]. The accuracy of the definition is checked by showing that known characterizations of classical ribbon knots generalize to higher dimensions in a natural way. Ribbon knots form an important part of this study of knot cobordisms, because they are knots which are cobordant to the unknot via the simplest cobordisms. Many of the applications we have of the more general theory will be to ribbon knots.

The relationship between ribbon knots and cobordisms to the unknot begins to surface in Chapter II. A constructive method is given there for determining the handle structure for the exterior of a cobordism, given information about how the submanifold is embedded. Examples which illustrate the technique are also given.

Chapter III begins a development of a handle theory for manifold pairs. Allowable "handle moves" are reviewed and developed.

Applications of the handle moves are given in Chapter IV. Other applications of results in this work are given, including examples of higher dimensional slice knots which are not ribbon knots, and an unknotting theorem for ribbon knots.

The dissertation is organized by chapters, divided into sections. Important statements (such as theorems, lemmas, equations, etc.) are numbered by the chapter (Roman) numeral followed by the ordinal number of the section in the chapter, followed by the ordinal number of the statement in the section. If a reference is to a statement within the same chapter, the chapter numeral is omitted. Figures are numbered beginning with each chapter. A reference to a figure in another chapter includes the chapter numeral as well as the figure number. Otherwise, the reference includes only the figure number.

The remainder of this section is devoted to establishing notation, definitions, and a few facts. Reference to these will be tacitly assumed whenever needed.

Notation

The category of smooth manifolds and smooth maps is used, unless otherwise stated. There will be occasion to use CW complexes, as well as the piecewise linear category. All manifolds will be oriented, and all maps between manifolds will be orientation preserving (we will not always remind the reader of this).

The symbol D^n_r will be used to denote the standard n-disk of radius r in whichever category is under discussion:

$$D^n_r = \begin{cases} \{(x_1,\ldots,x_n) \mid x_1^2+\ldots+x_n^2 \le r^2\} & \text{in the smooth category} \\ \{(x_1,\ldots,x_n) \mid \max\{x_1,\ldots,x_n\} \le r\} & \text{in the PL category.} \end{cases}$$

D^n will be used to denote the unit n-disk, D_1^n. Other symbols which will be used frequently include:

I	the unit interval $[0,1]$;
M^n	a manifold of dimension n;
int(M) or $\mathring{M}$	the interior of M;
∂M	the boundary of M;
cl(A)	the closure of the set A;
S^n	∂D^{n+1};
$\simeq$	homotopy equivalence;
$\cong$	an isomorphism in the category under discussion (diffeomorphism, PL homeomorphism, group isomorphism, etc.);
Z	the ring of integers;
$H_n(X;R)$	the n-dimensional homology R-module of X with coefficients in the ring R; if no ring is specified, the ring is Z;
#	connected sum.

Definitions

An *n-knot* $K = (S^{n+2}, fS^n)$ is a smooth, oriented, (n+2,n)-sphere pair, i.e., fS^n is a smooth submanifold of S^{n+2}, and f is a homeomorphism from S^n to the submanifold. The term "knot" will also be used to refer to the submanifold. Two n-knots, $K_1 = (S^{n+2}, f_1S^n)$ and $K_2 = (S^{n+2}, f_2S^n)$, are of the same *type*, or are *equivalent*, if there is an orientation preserving diffeomorphism $h: S^{n+2} \longrightarrow S^{n+2}$ such that $h \circ f_1 = f_2$ and such that $h|_{f_1}$ is also orientation preserving. If an n-knot is of the same type as the standard sphere pair (S^{n+2}, S^n), the knot is called the *unknot*, and the sphere pair is said to be *unknotted*. Also, a disk pair is said to be unknotted if it is homeomorphic to the standard pair. The study of knots originated with 1-knots, so these knots will be referred to as *classical knots*. The numerical notation used in this work for the classical knots is that of Reidemeister (see [26] or [27]).

An *n-link* $L = K_1 \cup \ldots \cup K_m$ *of m-components* is a disjoint union of m oriented and smoothly embedded n-spheres, $K_1, \ldots, K_m$, in S^{n+2}. Two n-links, L and L', are of the same *type* if there is an orientation preserving diffeomorphism $f: S^{n+2} \longrightarrow S^{n+2}$ such that for each i, $f(K_i) = K_i'$ and $f|_{K_i}$ is orientation preserving. An n-link L is *trivial* if there are m disjoint smoothly embedded (n+1)-disks, $B_1, \ldots, B_m$, such that $\partial B_i = K_i$ for each i.

An attempt to attach a group structure to the set of n-knots can be made using the connected sum operation; however, inverse elements fail to exist, so an abelian semi-group is all that results. To obtain a group structure, the set of n-knots can be partitioned into equivalence classes using the equivalence relation of knot cobordism. The set of equivalence classes then forms a group under the connected sum operation.

A *cobordism* is a triple of manifolds, (W, M_0, M_1), such that $\partial W = M_0 \cup M_1$. $(M\times I, M\times 0, M\times 1)$ is the *trival cobordism*, or *product cobordism*. Two n-knots, (S^{n+2}, f_1S^n) and (S^{n+2}, f_2S^n), are *cobordant* if there is a smooth oriented submanifold w of $S^{n+2}\times I$, with $\partial w = (f_1S^n\times 0) \cup (-f_2S^n\times 1)$ where w is homeomorphic to $S^n\times I$ and where $-f_2$ is the embedding obtained from $f_2 : S^n \longrightarrow S^{n+2}$ by reversing the orientations on both S^n and S^{n+2}. The pair $(S^{n+2}\times I, w)$ is called a *knot cobordism*. An n-knot is *null-cobordant*, or *slice* if it is cobordant to the unknot.

For a knot $K = (S^{n+2}, fS^n)$, the *complement* of K, $S^{n+2}-fS^n$, is an invariant of the knot type. As the complement is not compact, it is usually more convenient to study a deformation retract of the complement which is compact. This is called the exterior of the knot, and we define the term in a more general setting.

Let (M^m, N^n) be a smooth pair, and let $\nu(N)$ denote a tubular neighborhood of N in M (see Milnor [20]). The *exterior of N in M* is then defined as $cl(M-\nu(M))$. We will refer to exteriors of knots, disk pairs, and knot cobordisms throughout Chapters II, III, and IV.

The following facts will be used without reference:

1) For a knot (S^{n+2}, fS^n), $\nu(fS^n) \cong S^n\times D^2$ where the diifeomorphism takes S^n to fS^n (in other words, it is a "bundle" isomorphism). See Massey [19], for a proof.

2) For a disk pair (D^{n+3}, gD^{n+1}), $\nu(gD^{n+1}) \cong D^{n+1} \times D^2$ where the diffeomorphism takes D^{n+1} to gD^{n+1}. This follows since D^{n+1} is contractible.

3) Knot exteriors are homology circles.

<u>Proof</u>: We will show that knot complements are homology circles. Let (S^{n+2}, fS^n) be an n-knot. From the long exact sequence of the pair $(S^{n+2}, S^{n+2}-fS^n)$

$$\ldots \to H_{q+1}(S^{n+2}) \to H_{q+1}(S^{n+2}, S^{n+2}-fS^n) \to H_q(S^{n+2}-fS^n) \to H_q(S^{n+2}) \to \ldots$$

we conclude that $H_q(S^{n+2}-fS^n) = 0$ for $1 < q < n+1$ since $H_{q+1}(S^{n+2}, S^{n+2}-fS^n) \cong H^{n-q+1}(S^n)$ by Alexander duality. Also,

$$0 \to H_{n+2}(S^{n+2}-fS^n) \to \overset{Z}{\overset{\cong}{H_{n+2}(S^{n+2})}} \to \overset{Z}{\overset{\cong}{H_{n+2}(S^{n+2}, S^{n+2}-fS^n)}} \to$$

$$H_{n+1}(S^{n+2}-fS^n) \to H_{n+1}(S^{n+2}) \quad = \quad 0$$

shows that $H_q(S^{n+2}-fS^n) = 0$ for $q = n+1,\ n+2$ and

$$\overset{0}{\overset{\|}{H_2(S^{n+2})}} \to \overset{Z}{\overset{\cong\,(\text{Alexander Duality})}{H_2(S^{n+2}, S^{n+2}-fS^n)}} \to H_1(S^{n+2}-fS^n) \to \overset{0}{\overset{\|}{H_1(S^{n+2})}}$$

shows that $H_1(S^{n+2}, fS^n) \cong Z$. So $H_*(S^{n+2}-fS^n; Z) \cong H_*(S^1; Z)$. The Universal Coefficient Theorem then shows that for any ring R, $H_*(S^{n+2}-fS^n; R) \cong H_*(S^1; R)$.$\square$

4) Disc pair exteriors are homology circles.

<u>Proof</u>: Let (D^{n+3}, gD^{n+1}) be a disk pair, and $W = cl(D^{n+3} - (gD^{n+1}))$ where $\nu(gD^{n+1})$ is a tubular neighborhood of gD^{n+1} in D^{n+3}. The proof is then immediate from the Mayer-Vietoris sequence

$$\begin{array}{ccccccccc}
\cdots \to & \tilde{H}_{q+1}(D^{n+3}) & \xrightarrow{\partial *} & \tilde{H}_q(W \cap \nu(gD^{n+1})) & \to & \tilde{H}_q(W) \oplus \tilde{H}_q(\nu(gD^{n+1})) & \to & \tilde{H}_q(D^{n+3}) & \to \cdots \\
 & & & \wr\Vert & & \wr\Vert & & \wr\Vert & \\
 & 0 & & \tilde{H}_q(D^{n+1} \times S^1) & & \tilde{H}_q(W) \oplus H_q(D^{n+1} \times D^2) & & 0 & \\
 & & & \wr\Vert & & \wr\Vert & & & \\
 & & & \tilde{H}_q(S^1) & & \tilde{H}_q(W) & & . \square &
\end{array}$$

5) Exteriors of knot cobordisms are homology circles.

<u>Proof</u>: Let $(S^{n+2} \times I, w^{n+1})$ be a cobordism of n-knots, and Y the closed complement of a tubular neighborhood of w^{n+1} in $S^{n+2} \times I$. Also, let Y_0 denote the 0-level of Y, and D the cone over Y_0. Then $D \cup Y$ is a disk pair complement. The Mayer-Vietoris sequence

$$\cdots \longrightarrow H_q(Y_0) \longrightarrow H_q(D) \oplus H_q(Y) \longrightarrow H_q(D \cup Y) \longrightarrow \cdots$$

shows that Y is a homology S^1, since Y_0, D, and $D \cup Y$ are. $\square$

I. RIBBON KNOTS

§1. Introduction and Definitions

In this chapter, the classical idea of ribbon knots is discussed and generalized to higher dimensional knots, following Yanagawa [40] and Roseman [28]. The suitability of the generalized definition is verified by showing that the known characterizations of ribbon knots generalize to give characterizations in the higher dimensions as well. This process is culminated in Theorem III.3.3.

Ribbon knots arise naturally in a study of handle decompositions of knot cobordisms, as they are all cobordant to the unknot via a cobordism with a very nice handle decomposition (see §III.3). The techniques and results developed in this chapter will be used repeatedly in succeeding chapters.

Ribbon knots were first introduced by Fox [5] in 1962 for the case of classical knots. To translate his topological definition to the smooth category, we need some preliminary definitions. A smooth map $f:M^m \to N^n$ $(m \leq n)$ is called an *immersion* if the rank of the Jacobian of f is m everywhere. A 2-disk which is immersed in S^3 is called a singular 2-disk, and is called a *ribbon* if all the singularities are of the type shown in Figure 1, where the immersion identifies the arcs A'B' and A"B" with AB. The boundary of a ribbon is defined to be the image of the boundary of the 2-disk under the immersion. And finally, a knot is a *ribbon knot* if it is the boundary of a ribbon. Figure 2 shows the stevedore's knot, 6_1, and the ribbon it bounds.

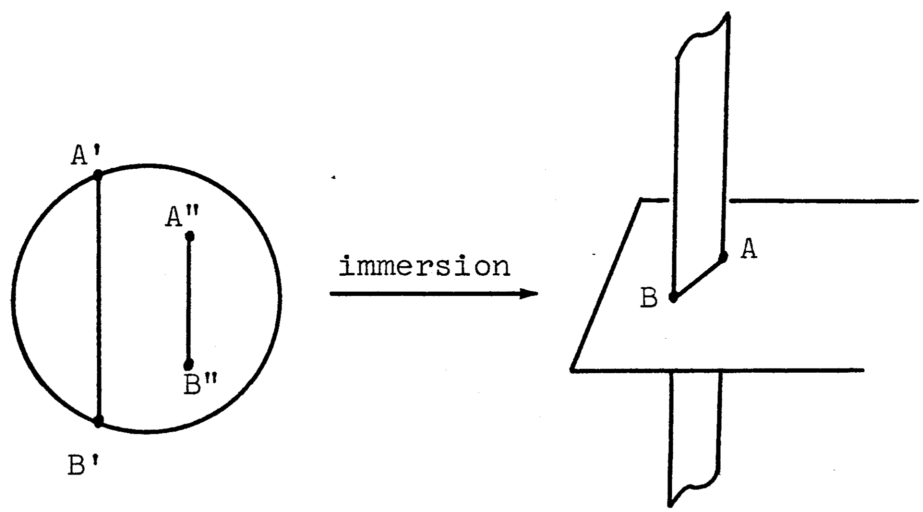


Figure 1

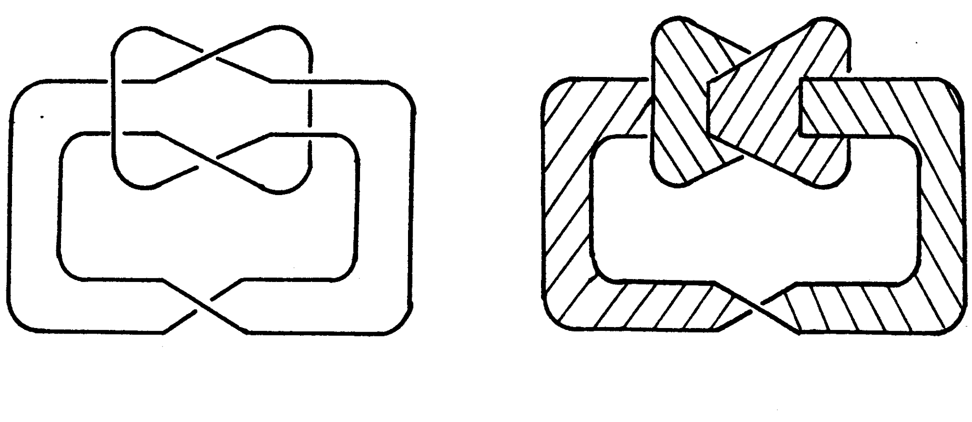


Figure 2

In 1964, Yajima [39] defined the concept *symmetric ribbon knot* for knots of S^2 in S^4. Five years later, Yanagawa [40] presented a definition for *ribbon 2-knots*, and gave several characterizations, including the equivalence of his definition with Yajima's.

In 1974, Roseman [28] gave two definitions of ribbon n-knots, both of which restrict to the classical definition when n=1, and the second of which agrees with Yanagawa's definition when n=2. The two definitions are:

Definition 1: An n-knot is a *ribbon n-knot* if it bounds an immersed (n+1)-disk $f: D^{n+1} \to S^{n+2}$, the singular set of which is a disjoint union $\cup\{P_i, Q_i \mid 1 \le i \le k\}$, where $P_i \cong Q_i$ is a compact connected n-manifold with non-empty boundary, $P_i \subset D^{n+1}$ is proper (i.e., $\partial P_i = P_i \cap \partial D^{n+1}$) and $Q_i \subset \mathrm{int}(D^{n+1})$ for each i, and $f(P_i) = f(Q_j)$ if and only if $i=j$.

Definition 2: Same as above, except that for each i, we require $P_i \cong Q_i \cong D^n$.

For n=1, both of these definitions coincide with Fox's.

Conjecture: For n≥2, there are knots which satisfy Definition 1, but not Definition 2.

In Chapter IV, we will show that there are higher dimensional slice knots which do not satisfy Definition 2 (e.g., the 2-twist-spun trefoil). We would conjecture that the 2-twist-spun trefoil does satisfy Definition 1.

From here on, Definition 2 will be used as the definition for ribbon n-knot. This is the more satisfactory definition, since the known characterizations of ribbon 1-knots can be generalized to ribbon n-knots under this definition. This process is begun in the next section.

§2. Fusions of Links

Let A, C_1, $C_2, \ldots, C_m$ be disjoint, unlinked, unknotted circles in S^3. Also, let $a_1, \ldots, a_m$ be disjoint arcs on A, and c_i an arc on C_i for $1 \le i \le m$. For each i, connect a_i with c_i by a band B_i in $\mathrm{cl}[S^3-(A \cup C_1 \cup \ldots \cup C_m \cup B_1 \cup \ldots \cup B_{i-1})]$, i.e., by a set $B_i \cong D^1 \times D^1$ such that $\partial D^1 \times \dot{D}^1$ corresponds to $a_i \cup c_i$. The set $k = (A \cup C_1 \cup \ldots \cup C_m \cup \partial B_1 \cup \ldots \cup \partial B_m) - (\mathring{a}_1 \cup \ldots \cup \mathring{a}_m \cup \mathring{c}_i \cup \ldots \cup \mathring{c}_m)$ is then a ribbon knot in S^3 (we will always assume that the "corners" are rounded in these types of constructions so that the resulting manifolds are smooth). Figure 3 shows the square knot presented in this form. In fact, Yajima [38] has shown that every ribbon 1-knot can be constructed in the above fashion, obtaining a characterization of ribbon 1-knots.

This characterization is generalized to ribbon 2-knots by Yanagawa [40]. In this section, the generalization is completed to ribbon n-knots.

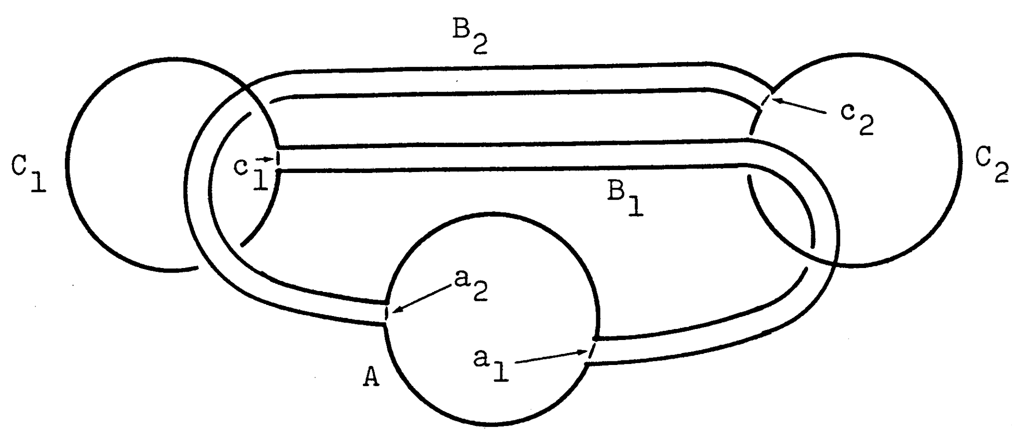


FIGURE 3

The idea of banding circles together is generalized in the following

Definition: Let $K = K_1 \cup \ldots \cup K_m \subset S^{n+2}$ be an n-link with m components ($m \geq 1$). If K' is obtained from K by piping (see [29], page 67) two distinct components of K together along an arc α with piping tube N, we say that K' is obtained from K by a *simple fusion* along N. Note that such a K' has m-1 components. After m-1 simple fusions are performed on K, the result is an n-knot, which we say is a *fusion* of the link K.

Note: The term "fusion" is used differently in each of Hosakawa [8], Yanagawa [40], and Suzuki [36].

Theorem 2.1: *An n-knot is a ribbon n-knot if and only if it is a fusion of a trivial n-link* ($n \geq 1$).

Proof: Let $f: D^{n+1} \to S^{n+2}$ be a ribbon immersion with singular set $\cup\{P_i \cup Q_i \mid 1 \leq i \leq k\}$ where $\partial P_i = P_i \cap \partial D^{n+1}$. Let $N_1, \ldots, N_k$ be disjoint tubular neighborhoods of $P_1, \ldots, P_k$, respectively, in D^{n+1}. Then the N_i's are (n+1)-disks, and $\partial[\mathrm{cl}(D^{n+1} - N_1 \cup \ldots \cup N_k)]$ consists of k+1 n-spheres, say $S_1, \ldots, S_{k+1}$. We can choose diffeomorphisms $\phi_i: D^1 \times D^n \to N_i$ such that $\phi_i(0 \times D^n) = P_i$ and such that $N_i \cap (S_1 \cup \ldots \cup S_{k+1}) = \phi_i(\partial D^1 \times D^n)$. Then each N_i is a pipe between two of the S_i's along the arc $\phi_i(D^1 \times 0)$.

Now, by construction, $f(S_1), \ldots, f(S_{k+1})$ forms a trivial n-link, and $f(\partial D^{n+1})$ is a fusion of this trivial

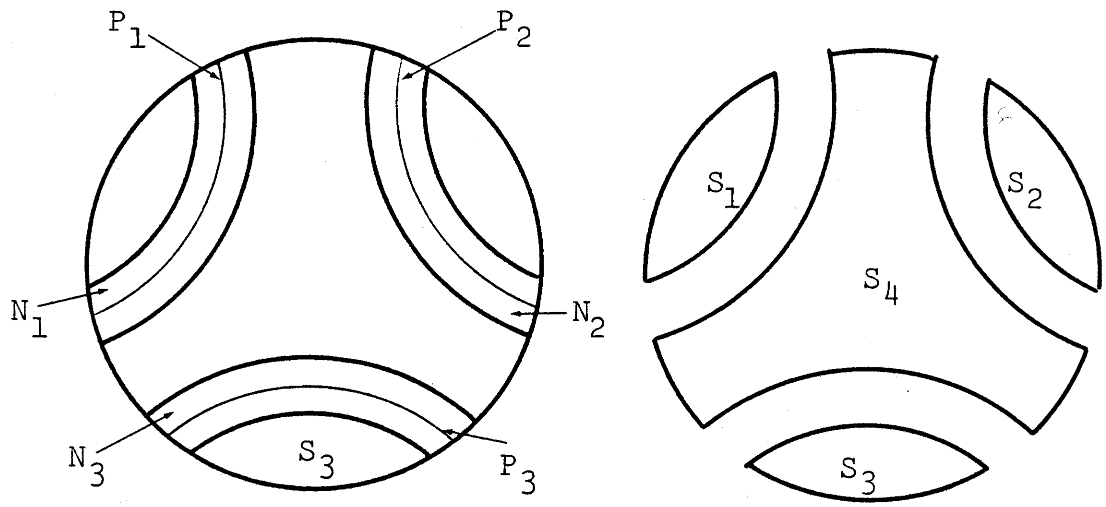


Figure 4

link along the pipes $f(N_i)$, $1 \le i \le k$ (see Figure 4).

Conversely, suppose a knot is a fusion of the trivial n-link $S_1, \ldots, S_{k+1}$ along the (n+1)-disks $N_1, \ldots, N_k$ in S^{n+2}. Let $B_1, \ldots, B_{k+1}$ be disjoint (n+1)-disks such that $\partial B_i = S_i$ for $1 \le i \le k+1$ (see Figure 5).

Without loss of generality, we may assume that any intersection of the form $B_i \cap N_j$ is an n-disk. To see this, first let $f_i : I \times D^n \to N_i$ be a diffeomorphism such that $f_i(\partial I \times D^n) = N_i \cap (S_1 \cup \ldots \cup S_{k+1})$ (see Figure 6). Then perform an ambient isotopy so that the sets $f_j(I \times 0)$ are transverse to the B_i's at each point of intersection. Now suppose $x \in f_j(I \times 0) \cap B_i$. Then there is an $\varepsilon > 0$ such that $f_j(I \times D^n_\varepsilon) \cap B_i$ is an n-disk, by transversality at x. Replace the pipe D_j by $f_j(I \times D^n_\varepsilon)$. After doing this for each point of intersection, the result is a ribbon immersed disk, the boundary of which is equivalent to the original knot.

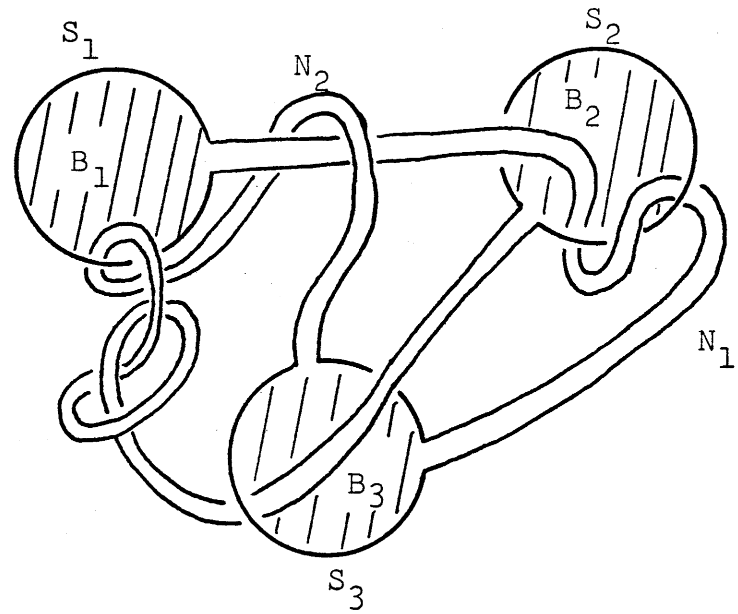


Figure 5

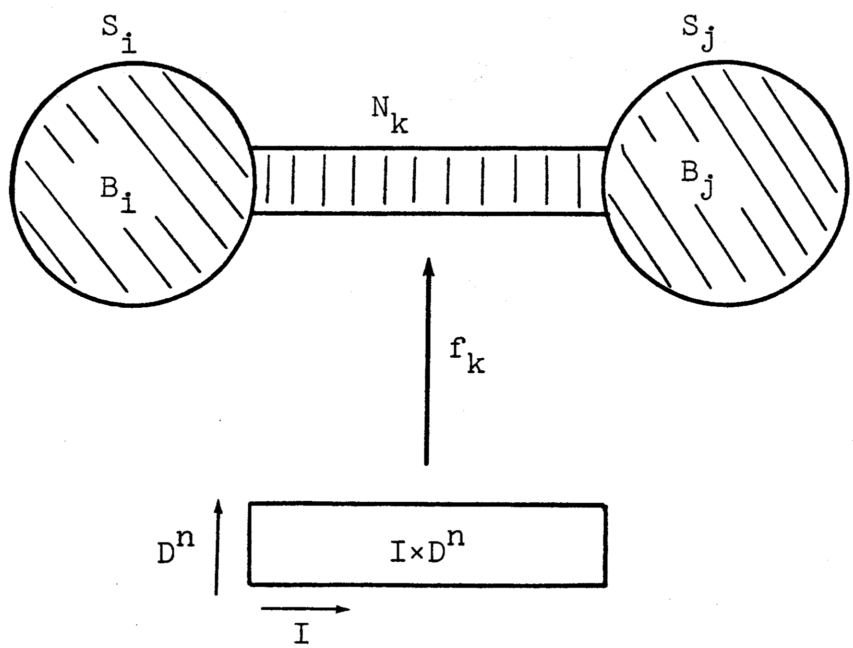


Figure 6

§3. Handle Decompositions of Seifert Manifolds of Ribbon Knots

We now turn to the problem of characterizing ribbon n-knots in terms of their Seifert manifolds. A *Seifert manifold* for (S^{n+2}, fS^n) is an orientable manifold $V^{n+1} \subset S^{n+2}$ such that $\partial V = fS^n$. By Levine [18], Seifert manifolds always exist. In this section, we study the handle structure of such Seifert manifolds.

Let M be an m-manifold with boundary, and h an m-disk such that $W \cap h \subset \partial W$, and suppose there is a diffeomorphism $f: D^p \times D^q \to h$ such that $f(\partial D^p \times D^q) = W \cap h$, where $p+q=m$. Then h is a *handle of index p* (a *p-handle*) attached to M. The symbol h^p will often be used to denote a p-handle. The notation $M \cup h^p \cup h^q$ means $(W \cup h^p) \cup h^q$, i.e., the handle h^q is attached to the manifold $W \cup h^p$.

Note that $M \cup h^p$ is a smooth m-manifold (after rounding corners). Slightly abusing notation, we will sometimes use the symbol h^p to denote the characteristic map, f, of the p-handle. So $h^p(D^p \times D^q) = h^p$. This will avoid introducing different symbols for the characteristic map in the future. The particular usage will be clear from the context.

Theorem 3.1: *An n-knot is a ribbon n-knot if and only if it has a Seifert manifold which is ambient isotopic to*

$$D^{n+1} \cup \{h_i^n \mid 1 \le i \le r\} \cup \{h_i^1 \mid 1 \le i \le r\}$$

where $D^{n+1} \cup \{h_i^n \mid 1 \le i \le r\}$ *is contained in an equatorial*

$(n+1)$-*sphere in* S^{n+2} $(n\geq 1)$.

Proof: Suppose an n-knot has a Seifert manifold in the above form. Then $\partial(D^{n+1}\cup\{h_i^n|1\leq i\leq r\})$ is a trivial n-link of $(r+1)$-components, and the knot is a fusion of this trivial n-link along the sets h_i^1 $(1\leq i\leq r)$. By Theorem 2.1, the knot is a ribbon n-knot.

Conversely, suppose (S^{n+2},fS^n) is a ribbon knot. By Theorem 2.1, fS^n is a fusion of a trivial n-link, say $S_1\cup\ldots\cup S_{k+1}$ along the piping tubes $N_1,\ldots,N_k$. As in the proof of Theorem 2.1, we may assume $fS^n=\partial[B_1\cup\ldots\cup B_{k+1}\cup N_1\cup \ldots\cup N_k]$ where the B_i's are pairwise disjoint $(n+1)$-disks, $\partial B_i=S_i$ for $1\leq i\leq k+1$, and the singularities of the immersed disk $A=B_1\cup\ldots\cup B_{k+1}\cup N_1\cup\ldots\cup N_k$ are all ribbon singularities. By an ambient isotopy of S^{n+2}, we can move the disks $B_1,\ldots,B_{k+1}$ so that they are contained in an equatorial $(n+1)$-sphere, S, in S^{n+2}. Also, by the transversality and tubular neighborhood argument used in Theorem 2.1, we can arrange it so that any intersection of a pipe N_i with S is a ribbon singularity. Now, let B be an $(n+1)$-disk in S which is large enough to contain the projection of the ribbon disk on S in its interior. Let α be an arc in B from B to B_{k+1} so that $\text{int}(\alpha)\subset\text{int}(B-A)$. Let T be a pipe in B along α from ∂B to fS^n (see figure 9(b)). Then the knot $\partial(B-(T\cup A))$ is equivalent to the original knot, kS^n, since $\partial(B-(T\cup A))$ is just the connected sum of fS^n with the unknot, ∂B (see figure 9(b)).

Suppose one of the pipes, say N_i, intersects the set $B-(T\cup B_1\cup\ldots\cup B_{k+1})$. This singularity can be removed by the following procedure, which is a generalization to higher dimensions of the "Dehn cuts" found in Papakyriakopoulos [25].

By smooth transversality, we may assume there is a diffeomorphism $d:D^{n+2}\longrightarrow S^{n+2}$ such that $d(0\times D^{n+1})$ is a tubular neighborhood of $E = N_j\cap(B-(T\cup B_1\cup\ldots\cup B_{k+1}))$ in N_i and $d(D^{n+1}\times 0)$ is a neighborhood in $B-(T\cup B_1\cup\ldots\cup B_{k+1})$ which contains E as a proper submanifold. Choose d so that the orientations of $d(0\times D^{n+1})$ and of $d(D^{n+1}\times 0)$ agree with the orientation of A.

Let $g:D^{n+1}\longrightarrow D^{n+1}$ be the diffeomorphism which is the identity on $0\times D^n\times 0$, and is defined as follows in the x_1x_{n+2}-plane:

$$g(x) = \begin{cases} x\cdot i \text{ if } x\in D^1_{\frac{1}{2}}\times 0\times D^1_{\frac{1}{2}} \\ x\cdot e^{i(1-|x|)\pi} \text{ if } x\in D^1\times 0\times D^1 - D^1_{\frac{1}{2}}\times 0\times D^1_{\frac{1}{2}} \end{cases}$$

thinking of the x_1x_{n+2}-plane as a complex plane with the x_{n+2}-axis as the imaginary axis. Figure 7 illustrates the action of g in the x_1x_{n+2}-plane.

Replace $d(0\times D^{n+1})$ with $d\circ g(0\times D^{n+1})$ and remove $\mathrm{int}(d(D^{n+1}_{\frac{1}{2}}\times 0))$ from the result (see Figure 8). This removes the ribbon singularity E without introducing any new singularities. So this process can be repeated until all the singularities are removed.

The result is a smooth (after rounding corners)

$(n+1)$-manifold, M, whose boundary is equivalent to the original knot. M is orientable by construction, and thus is a Seifert manifold for a knot in the class of fS^n, which we will still refer to as fS^n.

Let $C_1,\dots,C_r$ denote the $(n+1)$-disks $d(D_{1/2}^{n+1}\times 0)$ which were removed around each singularity. Then $cl(B-(T\cup B_1\cup\dots\cup B_{k+1}\cup C_1\cup\dots\cup C_r))$ is an $(n+1)$-disk with $k+r$ open disks removed from its interior, and so can be regarded as an $(n+1)$-disk with $k+r$ n-handles attached trivially. In fact, by construction, we have $D^{n+1}\cup\{h_i^n \mid 1\le i\le k+r\}\subset S$, an equatorial $(n+1)$-sphere in S^{n+2}.

The original 1-handles, which have now been split up into $k+r$ 1-handles, are attached to the set $D^{n+1}\cup\{h_i^n \mid 1\le i\le k+r\}$ to obtain $V = D^{n+1}\cup\{h_i^n \mid 1\le i\le k+r\}\cup\{h_i^1 \mid 1\le i\le k+r\}$ in the desired form. The proof is illustrated in Figures 9(a)-9(e).□

Since the n-handles are attached trivially in the above proof, we can find disjoint n-disks $\delta_i\subset\partial D^{n+1}$ $(1\le i\le k+r)$ such that $\partial\delta_i = S_i^{n-1}$ where S_i^{n-1} is the attaching sphere for h_i^n. Since ∂V is connected, it can easily be arranged so that the attaching set for each 1-handle (an $S^0\times D^{n+1}$) has one component in one of the δ_i's and the other component in $\partial D^{n+1} - \bigcup\{\delta_i \mid 1\le i\le k+r\}$. Then, of course, no δ_i can intersect two different 1-handles. Figures 9(e)-9(g) illustrate this process.

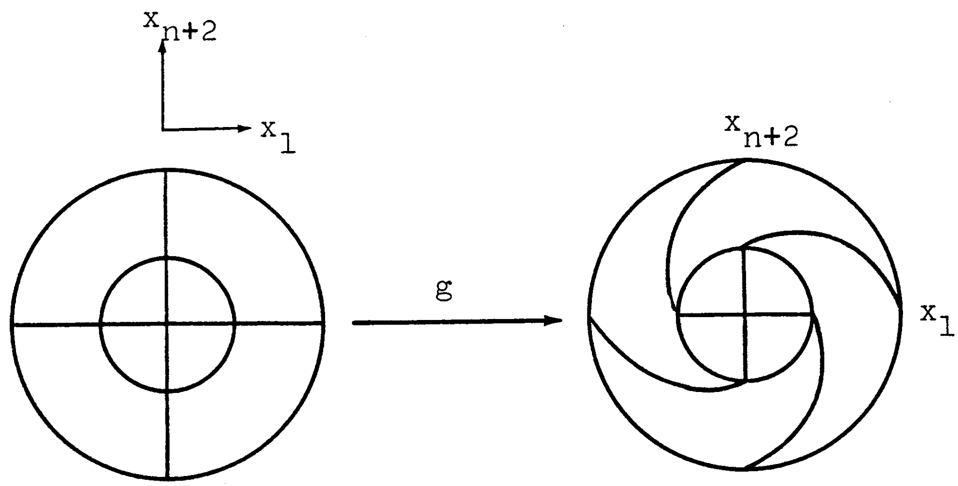


Figure 7

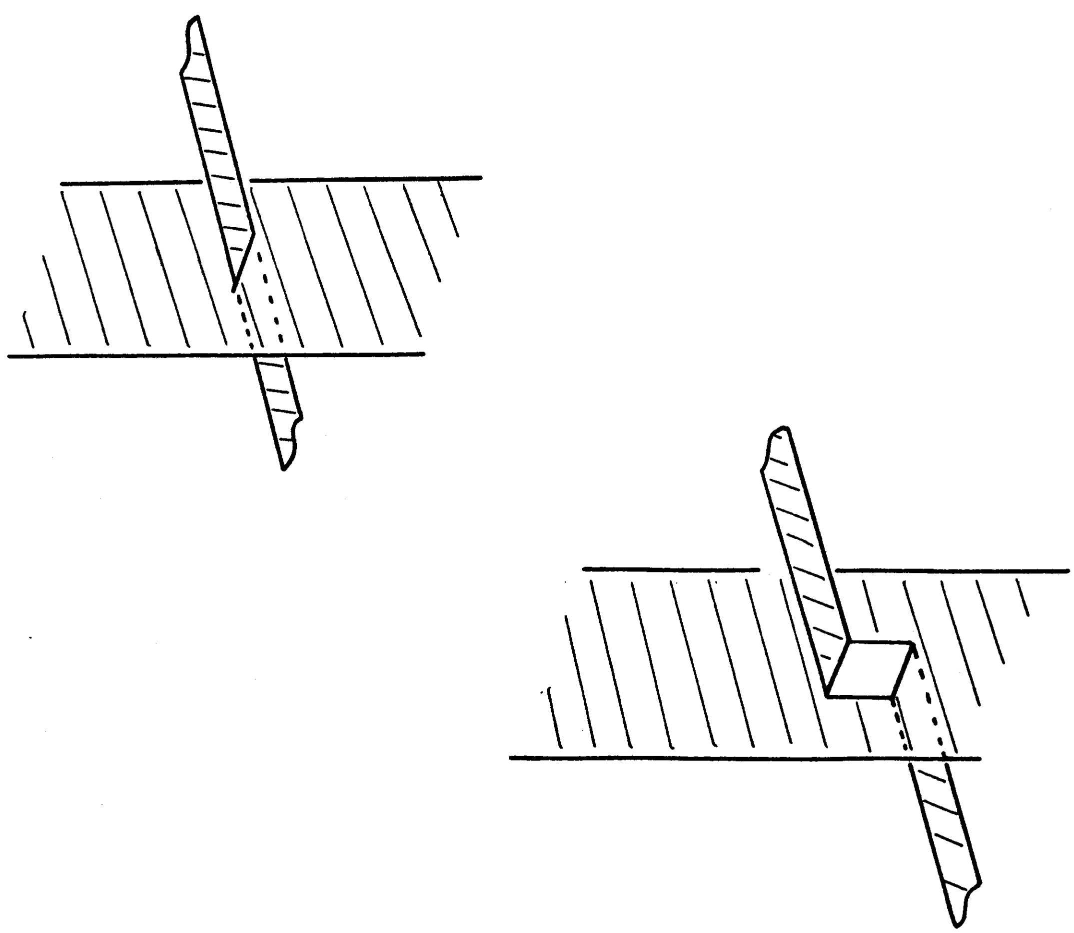
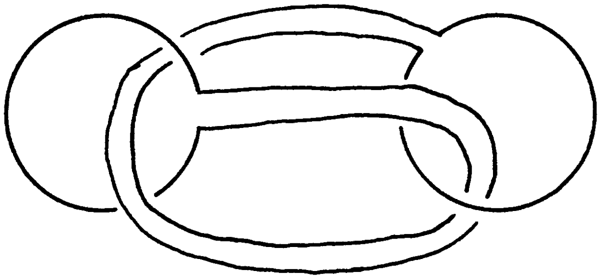
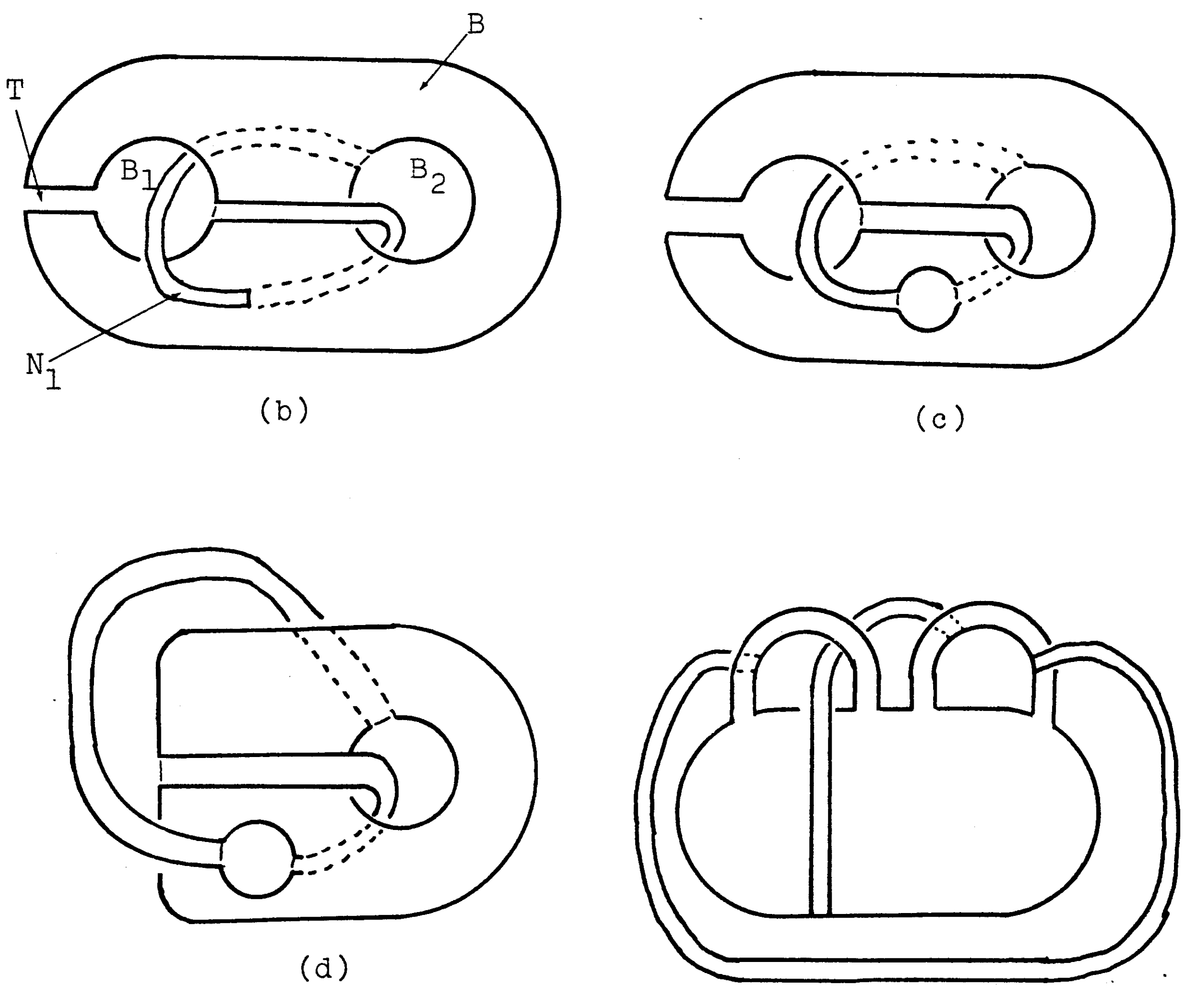


Figure 8



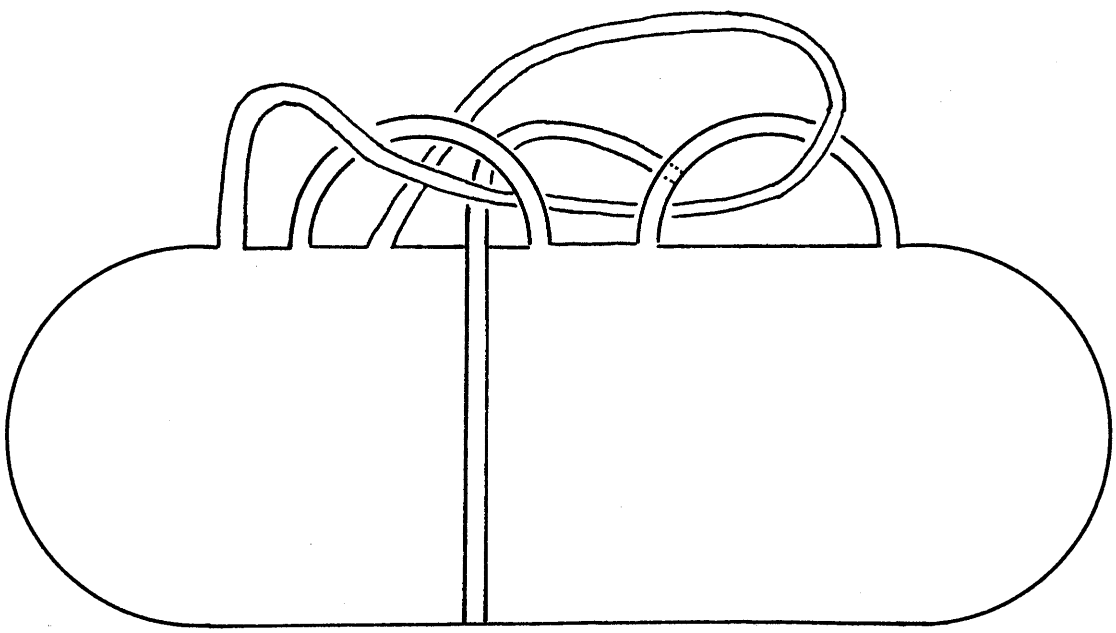
the square knot,
expressed as a
fusion of a
trivial link

(a)

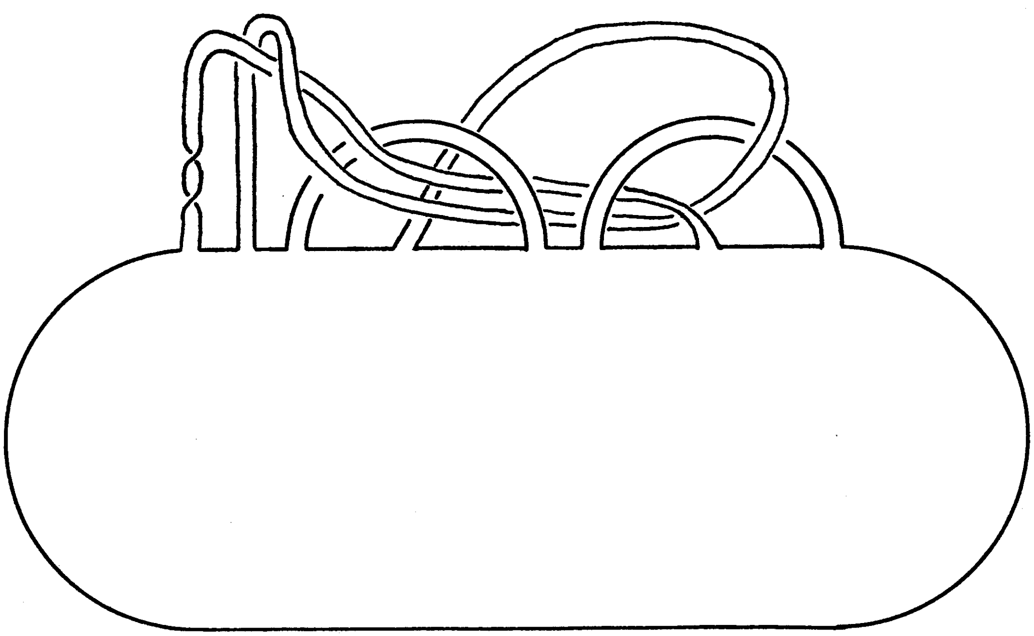


(b) (c) (d) (e)

Figure 9



(f)



(g)

Figure 9

Of course, not every Seifert surface of a ribbon knot need be of this special form, as the Seifert surface for the square knot depicted in Figure 10 shows. It is not clear, however, whether or not this Seifert surface is ambient isotopic to one in the special form.

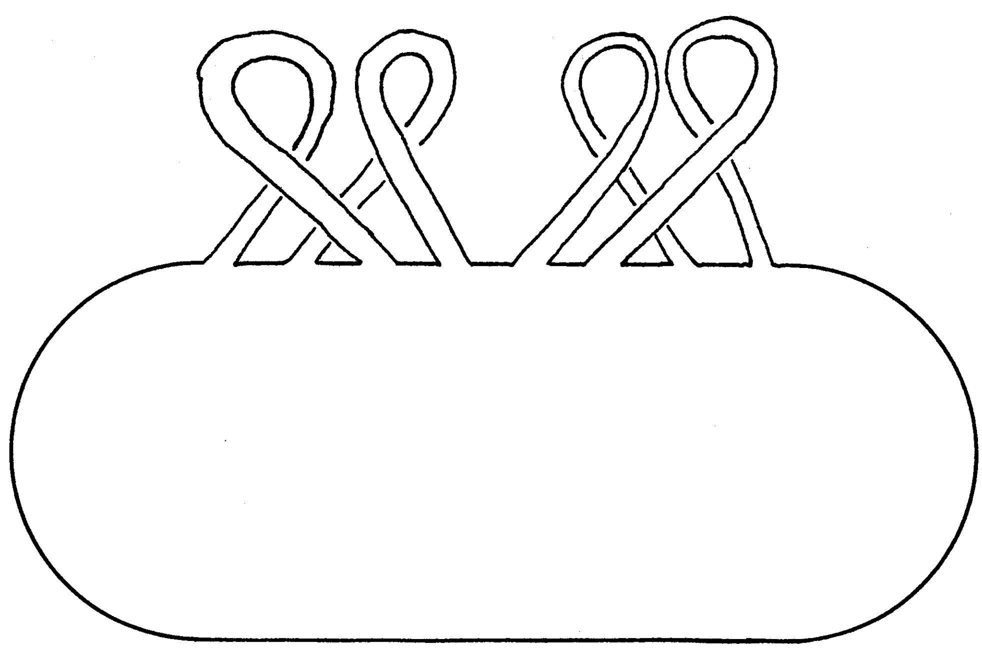


Figure 10

<u>Question</u>: Is every Seifert surface of a ribbon n-knot ambient isotopic to one in the form of Theorem 3.1?

In the classical case, the genus of the special Seifert surface constructed in Theorem 3.1 need not be minimal, either, since the surface of genus 1 shown in Figure 11 is a Seifert surface in the form of the theorem for the unknot -- the knot of genus 0.

<u>Question</u>: Given any ribbon knot, is there a Seifert surface for the knot in the form given in Theorem 3.1 which has minimal genus?

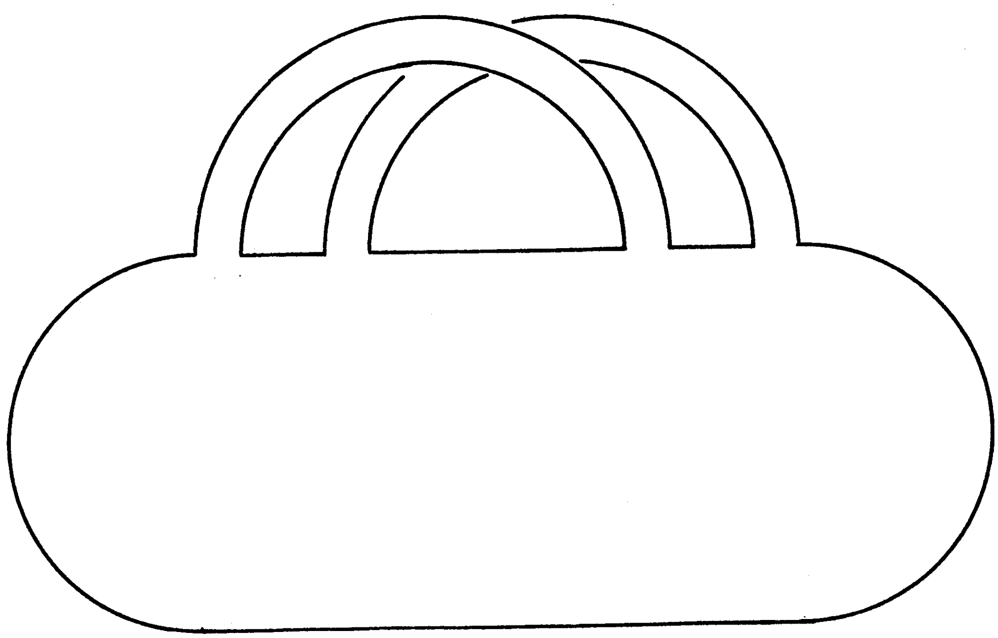


Figure 11

§4. Semi-Unknotted Manifolds

In [4], Fox gives a definition for a semi-unknotted surface, and shows that a knot is a ribbon 1-knot if and only if it bounds a semi-unknotted surface in S^3. In [40], Yanagawa gives a definition for a 3-manifold being semi-unknotted, and proves the analogous result for ribbon 2-knots. The following is a generalization of these definitions and theorems.

<u>Definitions</u>: Let M be an (orientable, as always) (n+1)-manifold in S^{n+2}. A collection of n-spheres $S_1^n, S_2^n, \ldots S_{2m}^n$ is called a *trivial system of n-spheres* in M if the following conditions are satisfied:

1) $\{S_i^n \mid 1 \le i \le 2m\}$ is a trivial n-link;

2) for each i, $1 \le i \le m$, there is an $N_i \cong S^n \times [0,1]$ in M such that $\partial N_i = S_i \cup S_{i+m}$ and $N_i \cap N_j = \emptyset$ for $i \neq j$; and

3) $M - \bigcup_{i=1}^{m} \mathrm{int}(N_i)$ is the closure of an (n+1)-disk with 2m smoothly embedded (n+1)-disks removed.

<u>Definition</u>: An (n+1)-manifold M in S^{n+2} is called *semi-unknotted* if

1) M is a disk; or

2) M has a trivial system of n-spheres.

We remark that if a manifold $M^{n+1} \subset S^{n+2}$ is semi-unknotted, then $M \cong \#(S^1 \times S^n) - B$ where B is an embedded (n+1)-disk in $\#(S^1 \times S^n)$. For let $N_1, \ldots, N_m$ be as in the definition of semi-unknotted. Then $M\text{-int}(N_1\cup\ldots\cup N_m)$ is an (n+1)-disk with the interiors of m disjoint (n+1)-disks removed from its interior. Attach a disk B to ∂M to form the smooth manifold $M' = D^{n+1} \underset{\partial B=\partial M}{\cup} M$. Then $M'\text{-int}(N_1\cup\ldots\cup N_m)$ is an (n+1)-sphere with m disjoint (n+1)-disks removed from its interior. When the N_i's are added on, we have $\#(S^1\times S^n)$, so $M \cong \#(S^1\times S^n)-B$, where B is an (n+1)-disk.

Theorem 4.1: *A knot* (S^{n+2},fS^n) *is a ribbon n-knot if and only if* $fS^n = \partial M \subset S^{n+2}$ *where is a semi-unknotted* (n+1)*-manifold* $(n\geq 1)$.

Proof: Suppose (S^{n+2},fS^n) is a ribbon n-knot. Then by Theorem 3.1, we may assume $fS^n=\partial M$ where $M = D^{n+1} \cup \{h_i^1 \mid 1\leq i\leq m\} \cup \{h_i^n \mid 1\leq i\leq m\}$ and, where the h_i^n's are attached trivially in an equatorial S^{n+1} in S^{n+2}. If $m=0$, then $fS^n=\partial D^{n+1}$ and we are done, so we assume $m\geq 1$ from here on. Since the h_i^n's are attached trivially, we can find disjoint n-disks $D_1,\ldots,D_m$ in $\text{int}(M)$ such that $\partial D_i = h_i^n(\partial D^n\times 0)$, so that $\{D_i\cup h_i^n(D^n\times 0) \mid 1\leq i\leq m\}$ is a trivial link. Let N_i be a tubular neighborhood of $D_i\cup h_i^n(D^n\times 0)$ in M which is sufficiently small so that $N_i\cap N_j=\phi$ for $i\neq j$ (see Figure 12). Then $\{\partial N_i \mid 1\leq i\leq m\}$ is a trivial link of 2m components. Furthermore, $M\text{-int}(N_1\cup\ldots\cup N_m)$ is an (n+1)-disk with 2m disks removed from its interior, which shows that the set $\{\partial N_i \mid 1 \le i \le m\}$ forms a trivial system of n-spheres in M.

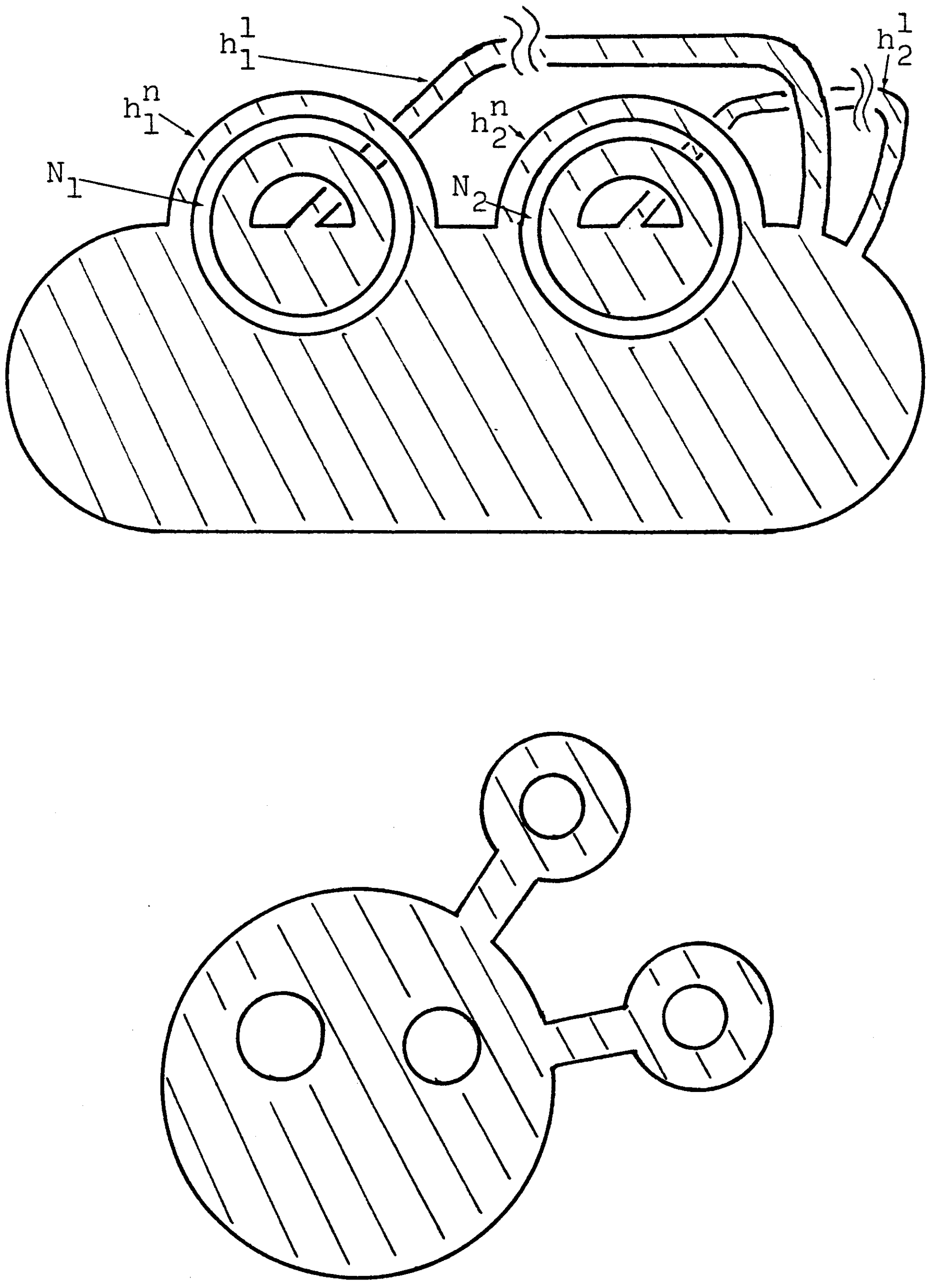


Figure 12

Conversely, suppose the knot (S^{n+2}, fS^n) bounds a semi-unknotted (n+1)-manifold M in S^{n+2}. Let $S_1, \ldots, S_{2m}$ and $N_1, \ldots, N_m$ be as in the definition of semi-unknotted manifold. Then $M' = M - \text{int}(N_1 \cup \ldots \cup N_m)$ is an (n+1)-disk with 2m (n+1)-disks removed from its interior. Let S be the boundary of an (n+1)-disk D_0 contained in the interior of M' and let $\{\alpha_i \mid 1 \le i \le 2m\}$ be a collection of disjoint proper arcs in M' such that α_i joins S_0 with S_i (see Figure 13). Then for sufficiently small open tubular neighborhoods P_i of the α_i's, $M' - (P_1 \cup \ldots \cup P_{2m} \cup \mathring{D}_0) \cong S^n \times I$.

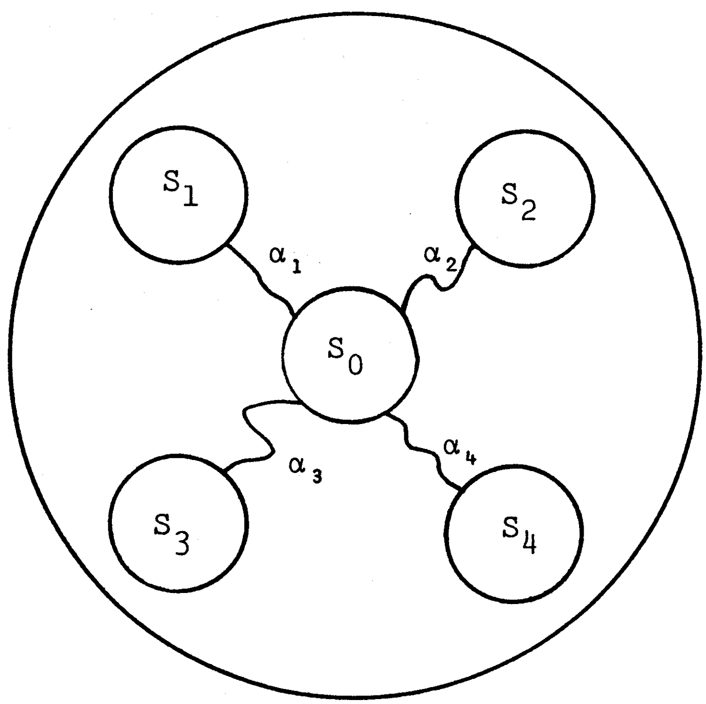


Figure 13

But then, one of the boundary n-spheres of $M'-(\mathring{D}_0 \cup P_1 \cup \ldots \cup P_{2m})$ is fS^n, and the other n-sphere is thus of the same type. But this n-sphere is a fusion of the trivial n-link $S_0, S_1, \ldots, S_{2m}$, hence is a ribbon knot by Theorem 2.1.□

This establishes three conditions which are equivalent to a knot being a ribbon n-knot. Three more conditions will be added in Chapter III to obtain Theorem III.3.3.

II. COBORDISMS OF KNOTS

§1. Introduction and Definitions

In this chapter, we begin our study of knot cobordisms. Theorem 2.1 shows how the handle structure of the exterior of a knot cobordism $(S^{n+2}\times I,w)$ can be calculated directly from the handle structure of w. The theorem is set up to apply to more general manifold pairs than knot cobordisms.

This theorem is applied in §3 to show that any ribbon knot is cobordant to the unknot via a cobordism built up with only 1- and 2-handles from the unknotted end. In Chapter III, we will obtain a partial converse to this theorem, and will obtain analogous results at the disk pair level. The constructive nature of the proof of Theorem 2.1 is illustrated by the calculation in §3 of such a 1-,2-handle decomposition for a null-cobordism of the knot 9_{46} (see [26]).

Let h denote a handle of index r attached to the manifold W^w. Thinking of h as the characteristic map from $D^r\times D^{w-r}$ to the handle, we call the set $h(D^r\times 0)$ the *core* of the handle and $h(0\times D^{w-r})$ the *cocore*. Also, $h(\partial D^r\times 0)$ is the *attaching sphere* and $h(0\times\partial D^{w-r})$ the *belt sphere*. The set $h(\partial D^r\times D^{w-r})$ is referred to as the *attaching set*.

If (W,M_0,M_1) is a cobordism, by an *r-handle on the* *cobordism* we will always mean an r-handle attached to W on M_1 -- i.e., the attaching set of the r-handle is contained in M_1. This then forms a new cobordism, $(W \cup h^r, M_0, M_1')$, where $M_1' = \partial(W \cup h^r) - M_0$. A *handle decomposition of W on* M_0 is a presentation

$$(1.1) \qquad W \cong M_0 \times I \cup h_1 \cup \ldots \cup h_m,$$

where each h_i is a handle on the cobordism $W_{i-1} \cong M_0 \times I \cup h_1 \cup \ldots \cup h_{i-1}$. Handle decompositions of cobordisms always exist -- see [9] or [29] for the piecewise linear case, and [22] for the smooth case.

Given a handle decomposition of W on M_0, as in (1.1) say, we can add a collar on M_1 to obtain

$$W \cong M_0 \times I \cup h_1 \cup \ldots \cup h_m \cup M_1 \times I.$$

In this setting, there is a corresponding *dual decomposition*

$$W \cong M_1 \times I \cup h_m^* \cup \ldots \cup h_1^* \cup M_0 \times I,$$

where $h_i = h_i^*$ as sets for each i, but the roles of the core and cocore of h_i are interchanged, yielding h_i^*.

Thus,

$$\text{index }(h_i^*) = w - \text{index}(h_i).$$

§2. Handle Decompositions of Exteriors of Submanifolds

We are looking for information about the handle structure of cobordism exteriors. The techniques we produce for obtaining this information apply in a much more general setting, however, and we develop this more general setting in this section.

For convenience, we will use the piecewise linear category. The proof works just as well for the smooth category, but there is a slight technical advantage to the piecewise linear approach. So, all manifolds here are PL and all maps between manifolds are PL

In [13], Kearton and Lickorish show that any locally flat PL embedding of a manifold M in $Q \times I$, where the dimension of the manifold Q is bigger than or equal to the dimension of M, is ambient isotopic to a critical level embedding. This is an embedding which, for some collared handle decomposition of M of the form

$$M = \text{collar} \cup \text{handle} \cup \text{collar} \cup \text{handle} \cup \ldots,$$

embeds each collar in the I direction of $Q \times I$, and each handle in a level of $Q \times I$. In the smooth category, this is analogous to requiring that the projection of M onto I is a Morse function for M. We will show (for $M \subset Q \times I$ codimension 2) how to obtain a handle decomposition of the exterior of M in $Q \times I$ if M is embedded as a critical level embedding. Each handle in the critical level decomposition of M will induce a handle in the exterior of M whose index is one larger than the index of the handle in M. The techniques used in the proof are modeled after techniques used by Vinogradov and Kushel'man [37].

Definition: Let M and Q be manifolds, and let a collared handle decomposition of M on $M_0 \times I$ ($M=\phi$ possible) be given. An embedding $f: M \to Q \times I$ is a *critical level embedding* for the given decomposition if f embeds the $i^{\underline{th}}$ handle in $Q \times t_i$ for some $t_i \in (0,1)$, and each collar of the form $X \times I$ is embedded as the product of an embedding $X \to Q$ and an order preserving embedding $I \to I$.

Theorem 2.1: *Suppose* M *and* Q *are compact manifolds of dimensions* n+1 *and* n+2, *respectively, and that* $M \cong M_0 \times I \cup \{h_i \mid 1 \le i \le p\}$ *is a handle decomposition of* M *on* M_0. *If* $f: M \to Q \times I$ *is a critical level embedding with respect to the given decomposition such that* $M_0 \times 0 \subset Q \times 0$ *and* $\partial M - (M_0 \times 0) = Q \times 1 \cap M$, *then the exterior of* M *in* $Q \times I$ *is homeomorphic to* $X_0 \times I \cup \{\hat{h}_i \mid 1 \le i \le p\}$ *where* X_0 *is the exterior of* $M_0 \times 0$ *in* $Q \times 0$, *and* index $\hat{h}_i = \text{index}(h_i) + 1$.

Proof: Since $M_0 \times 0 \subset Q \times 0$, and since the collars are embedded in an order preserving manner, there cannot be a handle embedded in $Q \times 0$. So, let $t \in (0,1)$ be the level of the lowest handle, say h_1, and suppose the index of h_1 is r. We may assume that h_1 is the only handle embedded at level t. Then h_1 is an (n+1)-disk attached to the n-manifold

$\overline{M}_t = M_t - h_1(\mathring{D}^r \times D^{n+1-r})$, where $M_t = M \cap Q\times t$ (see Figure 1). To describe the handle $\hat{h}_1$ in the exterior of M induced by h_1, we need to have a neighborhood of h_1 parameterized carefully. The following lemma accomplishes this.

Lemma 2.2: *There is a homeomorphism* $g: D_2^r \times D_2^{n+1-r} \times D^1 \longrightarrow N_h$ *where* N_h *is a regular neighborhood of the r-handle* h_1 *in* $Q\times t$ *such that:*

(i) N_h *is a subset of a regular neighborhood* N_t of M_t in $Q\times t$*;*

(ii) g *restricted to* $D_1^r \times D_1^{n+1-r}$ *is the characteristic map of* h_1*;*

(iii) $g(\partial D^r \times D^{n+1-r} \times 0) \subset \overline{M}_t$ *; and*

(iv) $g(D_2^r \times D_2^{n+1-r} \times \partial D^1 \cup \partial D_2^r \times D_2^{n+r} \times D^1) \subset \partial N_t$.

Proof (of lemma): The steps in the proof are illustrated in Figure 2. Let N" be a relative regular neighborhood of $h_1(D^r \times D^{n+r-1})$ (rel boundary) in $Q\times t$. We may assume that $N'' \cap M_t = h_1$. Then there is a homeomorphism $g'': D^r \times D^{n+1-r} \times D^1 \longrightarrow N''$ such that $g''(D^r \times D^{n+1-r} \times 0) = h^r$. Next, choose a collar C' of $g''(D^r \times \partial D^{n+1-r} \times D^1)$ in the manifold $cl(Q\times t - img'')$. This will induce a homeomorphism $g': D^r \times D_2^{n+1-r} \times D^1 \to N'$, where $N^1 = N'' \cup C$, and without loss of generality, we may assume $g'(\partial D^r \times D_2^{n+1-r} \times 0) = N' \cap \overline{M}_t$. Then, let C be a collar of $g''(\partial D^r \times D_2^{n+1-r} \times D^1)$ in the manifold $cl(Q\times t - img')$. Then there is a homeomorphism $g: D_2^r \times D_2^{n+1-r} \times D^1 \longrightarrow N_h$, where $N_h = N^1 \cup C$. By choosing C

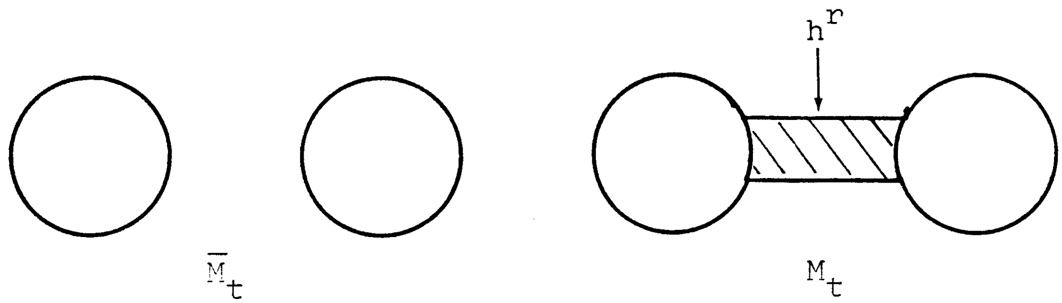


Figure 1

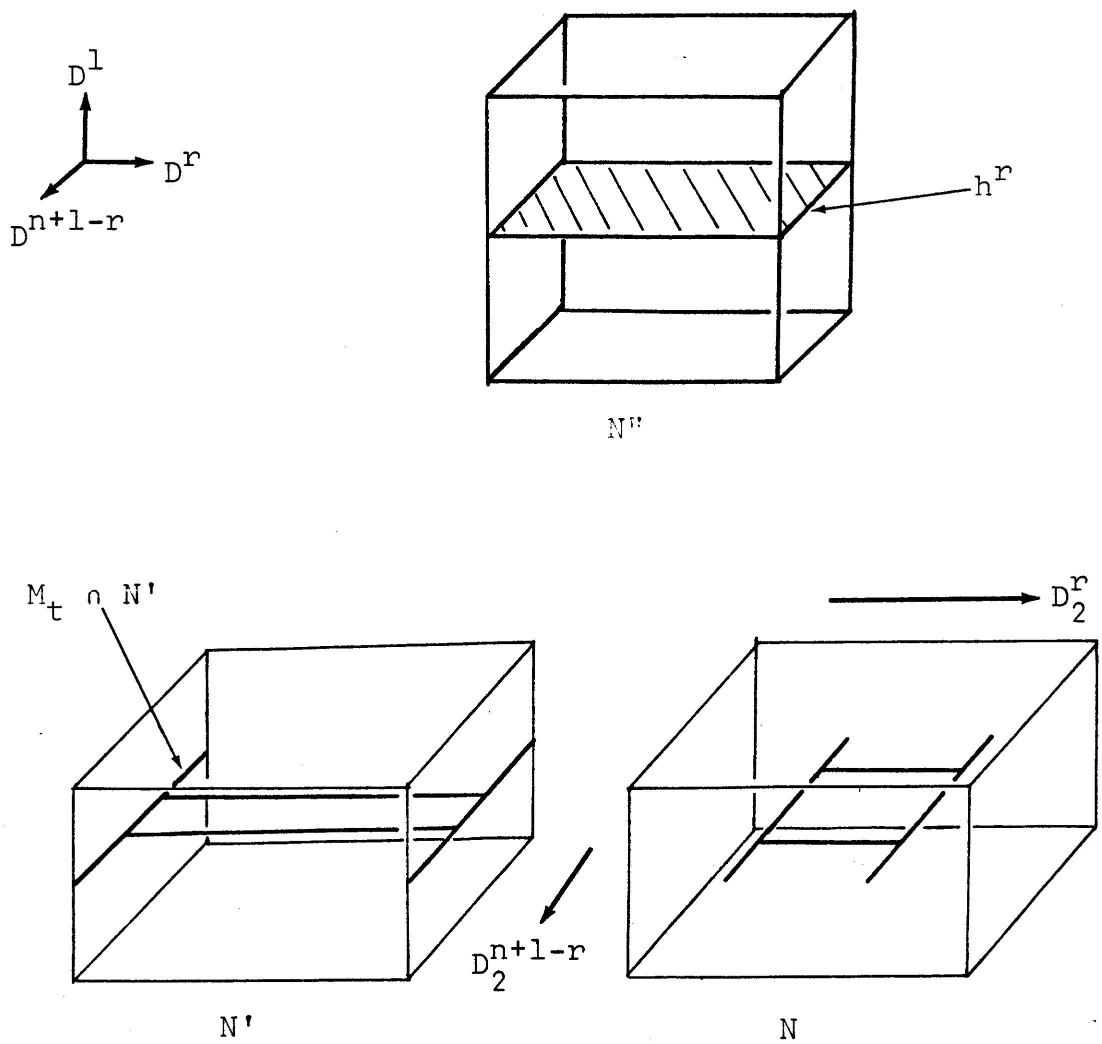


Figure 2

small enough, we can assume $N_h \cap \overline{M}_t = g''(\partial D_2^r \times D^{n+1-r} \times 0)$. N_h is then a regular neighborhood of h^r in $Q\times t$. N_h can easily be extended to a regular neighborhood of M_t so that condition (iv) is satisfied. By construction, (i) - (iii) hold.□

Returning now to the proof of the theorem, the regular neighborhood N_t can be extended to a regular neighborhood N of M in $Q\times I$. Let $X = \mathrm{cl}(Q\times I - N)$, the exterior of M. We will use X_S to denote $X \cap Q\times S$ where $S \subset I$. The following evident lemma will be used repeatedly.

Lemma 2.3: *Let* N *be a compact n-manifold,* M *an* (n+1)-*manifold, and suppose* $N \subset \partial M$. *Then the manifold* $W = M \cup N\times I$, *obtained by identifying* N *with* $N\times 0$, *is homeomorphic to* W.

By isotopy uniqueness of regular neighborhoods, we can assume there is an $\varepsilon>0$ such that $X_{[0,t-\varepsilon]} = X_0\times[0,t-\varepsilon]$, $X_{[t+\varepsilon,s-\varepsilon]} = X_{t+\varepsilon}\times[t+\varepsilon,t-\varepsilon]$, where s denotes the next critical level after t, and $X_{[t-\varepsilon,t+\varepsilon]} = X_t\times[t-\varepsilon,t+\varepsilon]$. Then the manifold $X_{[0,t]}$ is obtained from $X_{[0,t-\varepsilon]}$ by adding a collar on part of the boundary. So Lemma 2.3 applies to show $X_{[0,t]} \cong X_0\times[0,t-\varepsilon]$. Similarly, $X_{[t,s]} \cong X_{t+\varepsilon}\times[t+\varepsilon,s-\varepsilon]$. Using these homeomorphisms, we denote the subset of $X_0\times(t-\varepsilon)$ which corresponds to X_t by X_t^-, and the subset of $X_{t+\varepsilon}\times(t+\varepsilon)$ which corresponds to X_t

by X_t^+. Then, since

$$X_{[0,s]} = X_{[0,t]} \cup X_{[t,s]}, \text{ we have}$$

$$X_{[0,s]} \cong X_0 \times [0,t-\varepsilon] \underset{X_t^- = X_t^+}{\cup} X_{t+\varepsilon} \times [t+\varepsilon, s-\varepsilon].$$

We need to show that

(2.4) $$X_{t+\varepsilon} = X_t^+ \cup h^{r+1}, \text{ where}$$

the attaching set of h^{r+1} is contained in ∂N_t. Given this, we would have

$$\begin{aligned} X_{[t,s]} &\cong X_{t+\varepsilon} \times [t+\varepsilon, s-\varepsilon] \\ &= (X_t^+ \cup h^{r+1}) \times [t+\varepsilon, s-\varepsilon] \\ &= X_t^+ \times [t+\varepsilon, s-\varepsilon] \cup \hat{h}^{r+1}, \end{aligned}$$

where $\hat{h}^{r+1}$ denotes $h^r \times [t+\varepsilon, s-\varepsilon]$.

So,

$$\begin{aligned} X_{[0,s]} &\cong X_0 \times [0,t-\varepsilon] \underset{X_t^- = X_t^+}{\cup} X_{t+\varepsilon} \times [t+\varepsilon, s-\varepsilon] \\ &\cong X_0 \times [0,t-\varepsilon] \cup \left[X_t^+ \times [t+\varepsilon, s-\varepsilon] \cup \hat{h}^{r+1}\right] \end{aligned}$$

(2.5) $$\cong X_0 \times [0,t-\varepsilon] \cup \hat{h}^{r+1}, \text{ by Lemma 2.3.}$$

To see (2.4), we use lemma 2.1. Let S^+ denote the image of a set $S \subset X_t$ under the homeomorphism $X_{[t,s]} \cong X_{t+\varepsilon} \times I$. A regular neighborhood of $M_{t+\varepsilon}$ in $Q \times (t+\varepsilon)$ can be obtained from N_t^+ by deleting the set $g(D_2^r \times D_{\frac{1}{2}}^{n+1-r} \times D^1)^+$ from N_t^+. But this set is an $(r+1)$-handle attached along the r-sphere $g(D_2^r \times \partial D^1 \cup \partial D_2^r \times D^1)^+ \subset \partial N_t^+$ (see Figure 3).

Thus, $X_{t+\varepsilon} = X_t^+ \cup h^{r+1}$, as desired.

The same argument works for each critical level, so the proof of the theorem is complete. □

Our interest here is in codimension two. However, for codimension larger than two, say m, the proof of the previous theorem suggests that a critically embedded handle of index r would induce a handle of index r+m-1 in the exterior. The main adjustment necessary in the proof would be in Lemma 2.2.

This theorem, then, shows how a handle decomposition for the exterior of an embedding can be calculated from a critically embedded decomposition of a submanifold. An interesting problem is whether this procedure can be reversed -- i.e., given a handle decomposition, is it a decomposition of the exterior of some embedding, and if so, can the embedding be determined? Partial answers to this question will be obtained in Chapter III.

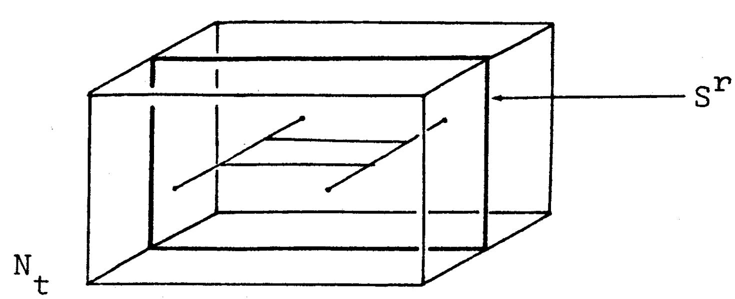
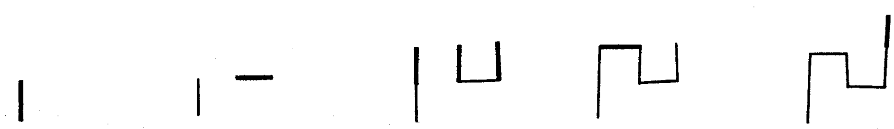


Figure 3

One of the advantages of carrying out the proof in the PL category is that the attaching sphere of a handle in the decomposition of the exterior is contained in a t-level of $Q\times I$ (see equation (2.5) in the proof). This is often an aid when attempting to geometrically compute and explicitly exhibit the attaching sphere. We illustrate the constructive nature of the proof in the following example.

Consider the embedding $f:D^1 \to D^2\times I$ illustrated in Figure 4. Then f is a critical level embedding with respect to the obvious decomposition of D^1 of the form

$$\text{collar} \cup \text{0-handle} \cup \text{collar} \cup \text{1-handle} \cup \text{collar}.$$



Let N be a regular neighborhood of $f(D^1)$, and let X denote the exterior of $f(D^1)$ in $D^2\times I$. Then, by the theorem, we know

$$\begin{aligned} X &\cong X_0\times I \cup h^1 \cup h^2 \\ &\cong (S^1\times I)\times I \cup h^1 \cup h^2. \end{aligned}$$

By examining the proof, we also know how the handles are attached (see Figure 5). From this, we see that X has the handle decomposition in Figure 6 (drawn without the 2-handle).

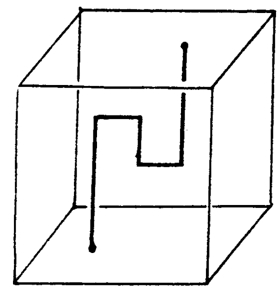


Figure 4

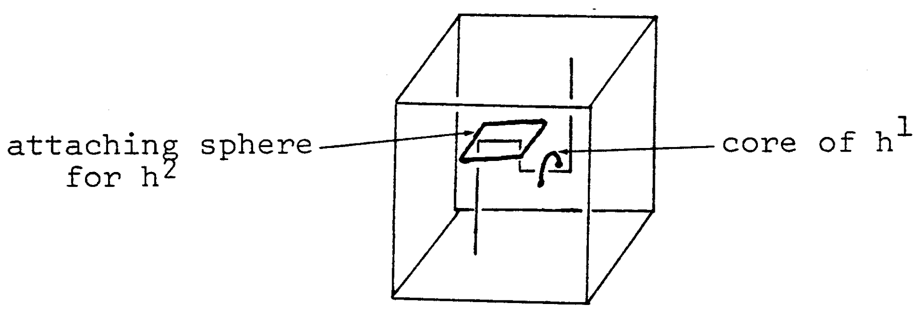


Figure 5

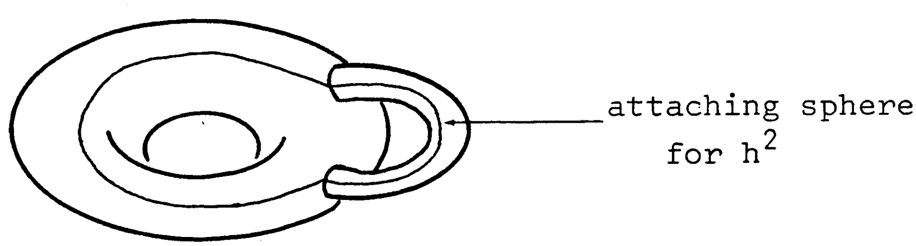


Figure 6

Even though only the attaching sphere for the 2-handle is exhibited in Figure 6, this uniquely determines the framing of the attaching set (up to ambient isotopy), since the attaching sphere is codimension one in the attaching set. When the codimension is larger, the framing of the attaching set has to be calculated from the geometry. Examples of this type are considered in the next section.

§3. Null Corbordisms of Ribbon Knots

The main purpose of this section is to begin the characterization of ribbon knots in terms of their null-cobordisms. We show that any ribbon knot has a null-cobordism, the exterior of which has a handle decomposition consisting entirely of 1- and 2-handles from the unknotted end. A partial converse to this is obtained in the next chapter.

To begin with, let us suppose that $K = (S^{n+2}, fS^n)$ is a ribbon knot. Theorem I.2.1 shows that fS^n is a fusion of a trivial n-link in S^{n+2} -- say $S_1,\ldots,S_{k+1}$ fused along the pipes $N_1,\ldots,N_k$. From this representation, it is easy to construct a critical level embedding $\bar{f}:D^{n+1} \longrightarrow S^{n+2}\times I$ such that $\bar{f}|_{\partial D^{n+1}} = f|_{S^n}$. Then, using Theorem 2.1, we will be able to compute a handle decomposition of the exterior of $\bar{f}(D^{n+1})$ in $S^{n+2}\times I$.

To construct the disk, first embed $f(S^n)\times[0,1/3]$ in $S^{n+2}\times I$ under inclusion. Then attach the pipes $N_i\times(1/3)$ to $f(S^n)\times[0,1/3]$. Next, attach

$$(S_1\cup\ldots\cup S_{k+1})\times[1/3,2/3]$$

to

$$f(S^n)\times[0,1/3] \cup N_1\times(1/3) \cup \ldots \cup N_k\times(1/3).$$

Now, let $B_1,\ldots,B_{k+1}$ be disjoint (n+1)-disks in S^{n+2} such that $\partial B_i = S_i$ for each i. Finally then, attach the disks $B_i\times(2/3)$ to the above set to obtain the disk $\bar{f}(D^{n+1})$, which is defined as

$f(S^n)\times[0,⅓] \cup (N_1\cup\ldots\cup N_k)\times⅓ \cup (S_1 \ldots S_{k+1})\times[⅓,⅔] \cup (B_1\cup\ldots\cup B_{k+1})\times⅔$.

Figure 7 illustrates the construction for the unknot presented as a fusion of a link with two components.

Note that each set $N\times(1/3)$ is actually an n-handle attached to the cobordism $fS^n\times[0,1/3]$, and each set $B_i\times(2/3)$ is an (n+1)-handle attached to the cobordism

(3.1) $L = fS^n\times[0,1/3] \cup (N_1\cup\ldots\cup N_k)\times(1/3) \cup (S_1\cup\ldots\cup S_{k+1})\times[1/3,2/3]$.

Thus the set $\bar{f}(D^{n+1})$ has been expressed in the form

(3.2) collar $\cup$ n-handles $\cup$ collar $\cup$ (n+1)-handles,

where the collar structure is embedded productwise along the I direction of $S^{n+2}\times I$, and each handle is in a t-level of $S^{n+2}\times I$ -- i.e., $\bar{f}$ is a critical level embedding with respect to the decomposition (3.2). This motivates the following definition.

Definition: An (n+1)-disk $fD^{n+1} \subset S^{n+2}\times I$ such that $f(\partial D^{n+1}) \subset S^{n+2}\times 0$ is called a *(PL) ribbon disk* if f is ambient isotopic to a critical level embedding with respect to a collared handle decomposition of D^{n+1} of the form

$S^n\times I \cup$ n-handles $\cup$ collar $\cup$ (n+1)-handles.

For the smooth case, we can go through the above construction, and then approximate the resulting disk (preserving ambient isotopy type) by a smoothly embedded

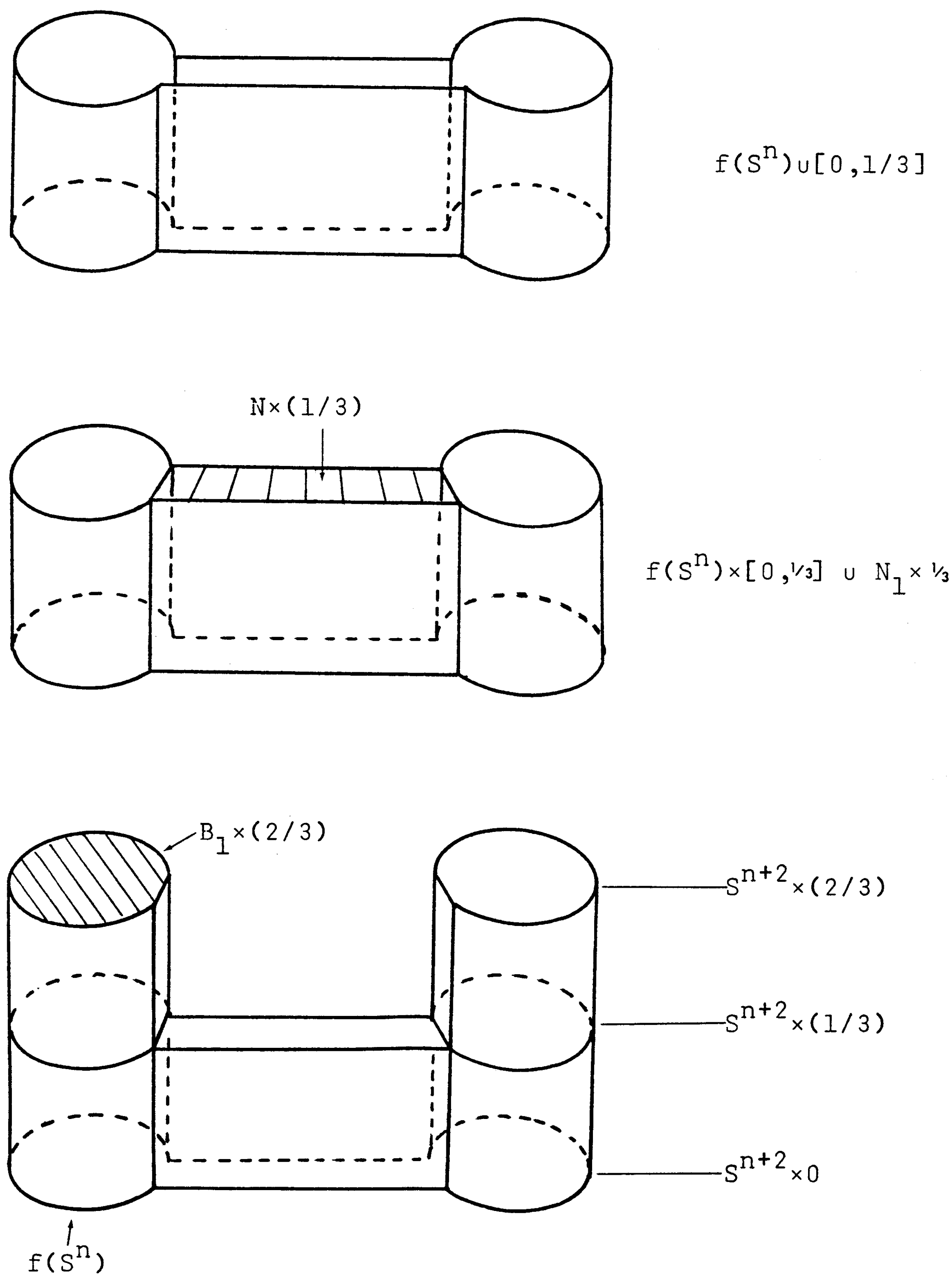

$f(S^n) \cup [0,1/3]$
$N \times (1/3)$
$f(S^n) \times [0,1/3] \cup N_1 \times 1/3$
$B_1 \times (2/3)$
$S^{n+2} \times (2/3)$
$S^{n+2} \times (1/3)$
$S^{n+2} \times 0$
$f(S^n)$

Figure 7

disk, say $g: D^{n+1} \longrightarrow S^{n+2} \times I$, such that projection onto the I factor of $S^{n+2} \times I$ is a Morse function for gD^{n+1} having critical points of index n and n+1 only (see [22] or [23]). For the classical case, Fox [3] uses this method to describe 2-disks which bound knots in 4-disks by exhibiting 3-dimensional cross-sections.

Thus, in the smooth category we have the following definition.

Definition: A smoothly embedded (n+1)-disk fD^{n+1} in $S^{n+2} \times I$ such $f(\partial D^{n+1}) \subset S^{n+2} \times 0$ is a *(smooth) ribbon disk* if the projection of fD^{n+1} onto I is a Morse function for fD^{n+1} having critical points of index n and n+1 only.

By the above construction, then, we have the following theorem.

Theorem 3.3: *Every ribbon knot* (S^{n+2}, fS^n) *bounds a ribbon disk in* $S^{n+2} \times I$ *where* $S^{n+2} = S^{n+2} \times 0$.

Now, given a ribbon disk, we can use Theorem 2.1 to find a handle decomposition of the exterior of the disk. Then, the exterior of the disk has a handle decomposition on the ribbon knot complement consisting of (n+1)- and (n+2)-handles only. The dual handle decomposition will then have handles of index 1 and 2, since $S^{n+2} \times I$ has dimension n+3. This yields the following result.

Theorem 3.4: *Given a ribbon* (n+1)-*disk in* $S^{n+2}\times I$, *there is is a handle decomposition on* $S^{n+2}\times 1$ *of the exterior of the ribbon disk consisting of* k+1 1-*handles,* k 2-*handles, and and no other handles.*

We can slightly alter the construction of the ribbon disk from the ribbon knot to produce a cobordism between the ribbon knot and the unknot. All we need to do is forget to add one of the (n+1)-handles on the submanifold. If we omit, say, the $(k+1)^{\underline{st}}$ one, we have

$$L \cup (B_1 \cup \ldots \cup B_k)\times(2/3) \cup S_{k+1}\times[2/3,1]$$

when L is defined by equation (3.1).

This produces the desired cobordism, and if we think of it as proceeding from the unknot to the ribbon knot, we can apply Theorem 2.1 (upside down) to obtain

Theorem 3.5: *Given any ribbon n-knot, there is a cobordism between it and the unknot which has a handle decomposition of its exterior consisting entirely of* k 1-*handles and* k 2-*handles from the unknotted end.*

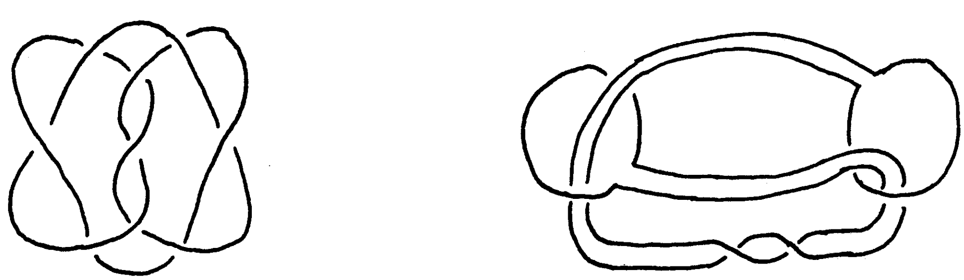
In the above construction, there actually were k+1 different choices for the omitted (n+1)-disk. We could have chosen to omit B_1, for example, instead of B_{k+1}. It is not clear if the various cobordisms obtained are equivalent as manifold pairs.

Question: Are different cobordisms obtained when different (n+1)-disks are omitted from a ribbon disk to obtain a cobordism?

It is worthwhile to consider this theorem from a slightly different point of view. Given a ribbon knot $K = (S^{n+2}, fS^n)$, we construct a cobordism between K and the unknot as above. We can then cap off the unknotted end of the cobordism with the unknotted disk pair, to obtain an (n+3,n+1)-disk pair whose boundary is the given ribbon sphere pair, K.

Another way of achieving the same result would be to choose some large disk $D_r^{n+2} \subset S^{n+2}$ such that $fS^n \subset D_r^{n+2}$. Then a ribbon n-disk can be constructed in $D_r^{n+2} \times I$ using the method described earlier. The (n+3,n+1)-disk pair produced here would be homeomorphic (as pairs) to the pair in the previous paragraph. We will make use of these observations in §III.3.

It is instructive at this point to consider an example. Figure 8(a) illustrates the classical ribbon knot, 9_{46}. After some manipulation, 9_{46} can be expressed as a fusion of a trivial link, as shown in Figure 8(b).



(a) (b)

Figure 8

Consider the representation of 9_{46} in Figure 8(b) as a subset of D^3, then we can construct a critical level embedding of a 2-disk in $D^3 \times I$ by the method discussed earlier in this section. Theorem 2.1 can then be used to calculate a handle decomposition of the exterior of the 2-disk in $D^3 \times I$, starting at the top. This would yield a 4-disk, two 1-handles, and a 2-handle. Up to homeomorphism type, there is only one way to attach the 1-handles. The only thing to determine then is how the 2-handle is attached.

From equation (2.5) in the proof of Theorem 2.1, it follows that

$$X \times [0,2/3] \cong X \times [0,1/3] \cup \hat{h}^2.$$

In particular, the attaching sphere for $\hat{h}^2$, say $\hat{U}$, is contained in $S^3 \times (1/3)$. But $\hat{h}^2$ is obtained from a 3-dimensional handle of index 2, h^2, by crossing with an interval. So the attaching curve of h^2, say U, together with $\hat{U}$, bounds an annulus, $A \subset S^3 \times (1/3)$, which comes from the product structure on $\hat{h}^2$ (see Figure 9). But Lemma 2.2 asserts that U bounds a 2-disk $B \subset S^3 \times (1/3)$ such that $B \cup A$ is a 2-disk in $S^3 \times (1/3)$. This shows that U and $\hat{U}$ are unlinked when viewed from $S^3 \times (1/3)$. So, not only do we have the attaching sphere for $\hat{h}^2$, we know that the attaching sphere for h^2 is a "push off" of $\hat{U}$ in the product structure of $\hat{h}^2$.

With this in mind, we can compute the attaching curve for the 2-handle. Figure 10 shows the critically embedded

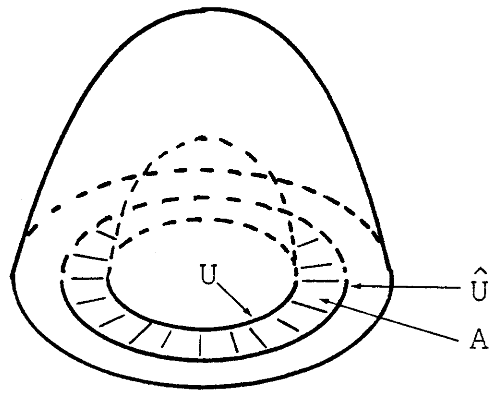


Figure 9

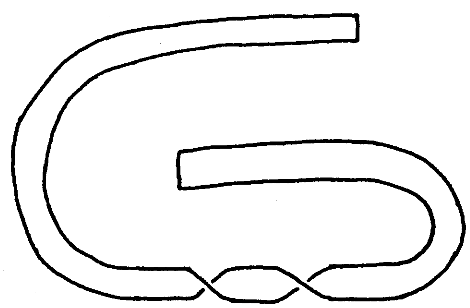


Figure 10

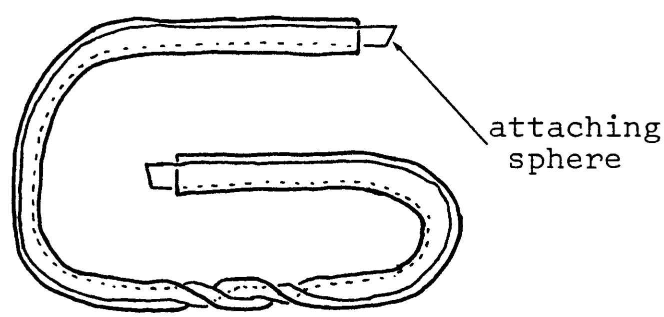


Figure 11

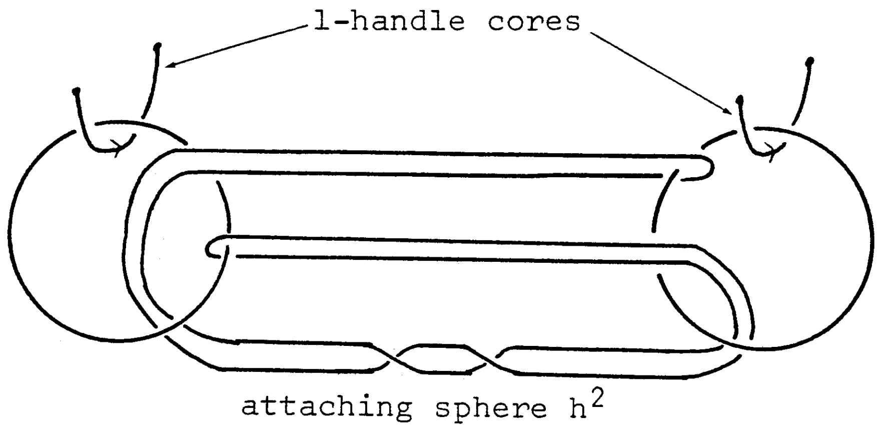


Figure 12

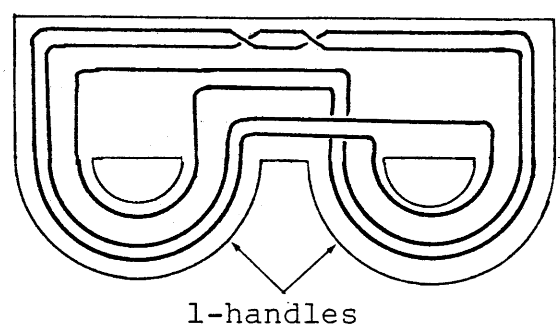


Figure 13

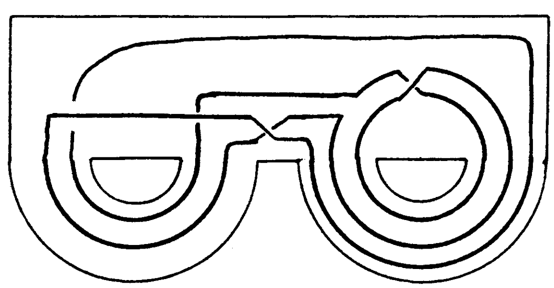


Figure 14

1-handle on the submanifold. By Lemma 2.2, the attaching curve for h^2 is the 1-sphere which bounds the normal 2-disk indicated in Figure 11. By the above observations, then, the attaching sphere for h^2 is the unlinked push-off of this sphere.

The relationship between the attaching sphere of the 2-handle and the two 1-handles is illustrated schematically in Figure 12. This translates to the handle diagram illustrated in Figure 13, being careful that all the orientations are preserved using the correct crossovers.

In [10] and [34], Sumners obtains a handle diagram for the exterior of a disk pair which bounds the knot 9_{46}. His handle diagram is illustrated in Figure 14. Notice that the attaching spheres of the 2-handles in Figures 13 and 14 are not isotopic. So it is not clear if the two handle presentations are homeomorphic, or if the disk pairs from which they come are equivalent.

To be able to solve problems of this type, a handle calculus for pairs needs to be developed. This is the main topic of the next chapter.

III. THE HANDLE CALCULUS FOR PAIRS

§1. Handle Presentations of Manifold Pairs

In this chapter, we point out how the well-known theory of handle moves on cobordisms can be applied to produce a theory of handle moves for certain manifold pairs. These techniques are developed in §2. In §3, the techniques are applied to ball pairs which bound ribbon knots, extending the results of Chapter II. Some applications and examples then follow. The handle calculus here is an analogue to that of Kirby [15].

To facilitate the development of these handle moves, we will use the concept of a "cobordism with boundary." By a *cobordism with boundary*, we mean a compact $(m+1)$-manifold W together with two disjoint compact m-dimensional submanifolds with boundary, M_0 and M_1, contained in ∂W. An *r-handle on a cobordism with boundary* will always be assumed to be attached to $\operatorname{int} M_1$, and will be denoted $W \cup h^r$.

<u>Definition</u>: 1) A manifold pair (M,Q) is called *proper* if $Q \cap \partial M = \partial Q$.

2) Let (M,Q) be a compact proper manifold pair. A *handle presentation of the pair* (M,Q) is a homeomorphism $(M,Q) \cong (N,Q) \cup$ handles, where N is a regular neighborhood (rel ∂Q) of Q in M, and where the attaching set of

each handle misses Q. When we write $(N,Q) \cup$ handles, we will assume the handles are attached away from Q.

Given a manifold pair, if an open regular neighborhood of the submanifold is removed, a compact manifold remains which can be viewed as a cobordism with boundary. This allows techniques used in the handle theory of cobordisms to be transferred to manifold pairs. The proof of the next theorem illustrates this procedure.

Theorem: *Let* (M,Q) *be a compact proper manifold pair. Then* (M,Q) *has a handle presentation.*

Proof: Let N be a regular neighborhood of Q in M. Also, let $W = cl(M-N)$, and $M_0 = W \cap N$. Choose a collar V on ∂M_0 in $cl(\partial W - M_0)$, and define $M_1 = cl(\partial W - (M_0 \cup V))$. Then W is a cobordism with boundary between M_0 and M_1 (see Figure 1).

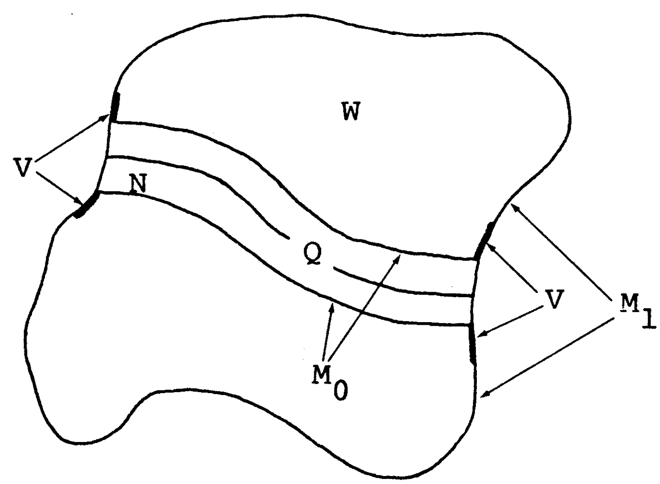


Figure 1

Let C be a collar on M_0 in W. By isotopy uniqueness of collars, we may assume that the product structure on the collar C extends the product structure on the collar V. Then there is a handle decomposition of W on M_0 rel V:

$$W = C \cup \text{handles} \quad \text{(see Figure 2)}.$$

Plugging the pair (N,Q) back in by the identity map, we have

$$(M,Q) = (N,Q) \cup C \cup \text{handles}$$

$$\cong (N,Q) \cup \text{handles}.$$

Thus, any proper manifold pair (M,Q) has a handle presentation.

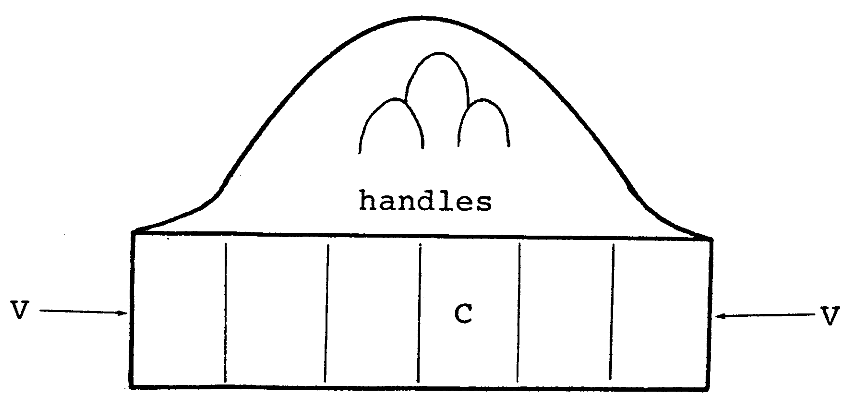


Figure 2

§2. The Handle Moves

Let W^m be a cobordism with boundary between M_0 and M_1, and let $V = cl(\partial W-(M_1 \cup M_0))$. We will review the handle moves in the following lemmas. Proofs to these lemmas are contained in [29].

Lemma 2.1: *Let* $\alpha,\beta:\partial D^r \times D^{n+2-r} \longrightarrow \mathrm{int}M_1$ *be ambient isotopic embeddings in* M_1*; then there is a homeomorphism* $h:W\underset{\alpha}{\cup}h^r \xrightarrow[\mathrm{rel}\,V]{} W\underset{\beta}{\cup}h^r$ *which leaves* V *pointwise fixed.*

Lemma 2.2 (Reordering lemma): *Let* $W' = W \cup h^r \cup h^s$ *where* h^r *is attached to* $\mathrm{int}M_1$, h^s *is attached to* $\mathrm{int}M_2 = \partial(W\cup h^r)-(V\cup M_0)$, *and* $s\leq r$. *Then* $W' \cong W \cup h^s \cup h^r$ *where* $h^s \cap h^r = \phi$, *and the homeomorphism leaves* V *pointwise fixed.*

Definition: Two handles, h^r and h^{r+1}, are called *complementary* if the attaching sphere of h^{r+1} intersects the belt sphere of h^r transversely at a single point.

Lemma 2.3 (Cancellation lemma): *Suppose* $W' = W \cup h^r \cup h^{r+1}$ *where* h^r *and* h^{r+1} *are complementary. Then there is a homeomorphism* $h:W'\longrightarrow W$ *which is the identity outside an arbitrary neighborhood of* $h^r \cup h^{r+1}$ *(and hence, a homeomorphism rel* V*).*

Lemma 2.4 (Adding lemma): *Let* $W' = W \underset{\alpha}{\cup} h_1^r \underset{\beta}{\cup} h_2^r$ *such that* $\mathrm{im}\alpha \cap \mathrm{im}\beta = \phi$, $n-r \geq 2$, *and* $r \geq 2$. *Also, let* γ *be* *the attaching map obtained as follows: choose an* $x \in \partial D^{n+2-r}$*, and pipe (in* $\mathrm{int}M_1$*) the attaching sphere of* h_2 *to* $c(\partial D^r \times x \times 1)$*, where* c *is a collar of the boundary of the attaching set of* h_1 *in* $\partial(\mathrm{cl}[W-(M_0 \cup V \cup h_1 \cup h_2)])$*; the result of the piping collapses to the attaching sphere of* h_2*, and this collapse defines an isotopy which takes* $\mathrm{im}\beta$ *to* $\mathrm{im}\gamma$*. Then* $W' \cong W \underset{\alpha}{\cup} h_1^r \underset{\gamma}{\cup} h_2^r$ *(rel* V*) (see Figure 3).*

<u>Remark 2.5</u>: To apply these handle moves to proper manifold pairs of the form (M^{n+2}, fD^n), we obtain a cobordism with boundary from the pair as in §1. As before, let N be a regular neighborhood of fD^n in M^{n+2},
$W = \mathrm{cl}(M-N)$, $M_0 = W \cap N$, V a collar on ∂M_0 in $\mathrm{cl}(\partial M - M_0)$, and $M_1 = \mathrm{cl}(\partial W-(M_0 \cup V))$. Then W is a cobordism with boundary between M_0 and M_1 (see Figure 1).

Now, all the isotopies and homeomorphisms in Lemmas 2.1 - 2.4 were done leaving $V \cup M_0$ pointwise fixed. Thus, if we perform a handle move on the cobordism W to obtain W', $W \cup N$ will be homeomorphic to $W' \cup N$ since
$W \cong W'$ (rel $V \cup M_0$). Also, we can choose the homeomorphism $W \cup N \cong W' \cup N$ to be the identity on N, so that the pairs (M,Q) and (M',Q) are homeomorphic, where $M' = \partial(W' \cup N)$. So, each of the Lemma's 2.1 - 2.4 induces a "handle move" on a handle presentation of a manifold pair.

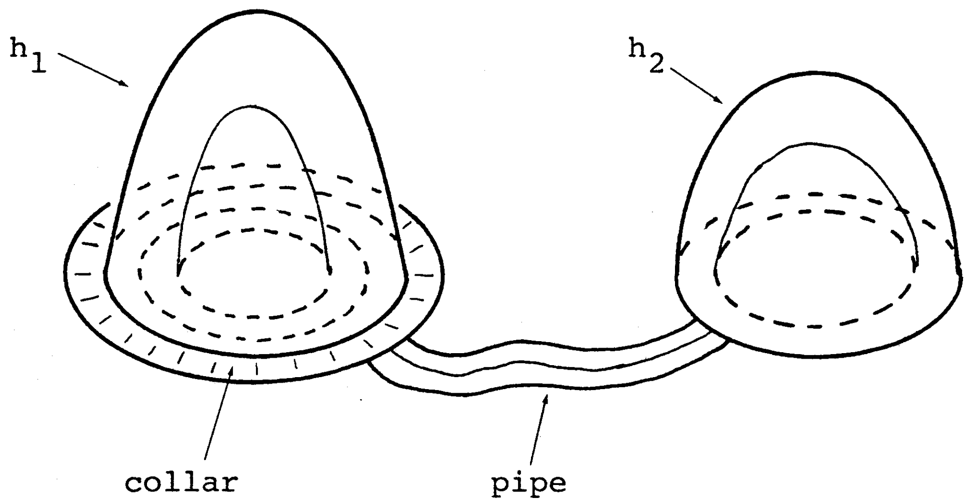


Figure 3

§3. Handle Moves on Disk Pairs with 1- and 2-handles

Theorem II.3.5 shows that any ribbon n-knot is null-cobordant by means of a cobordism built up with only 1- and 2-handles from the unknotted end. If the unknotted end of the cobordism is capped off with the unknotted disk pair, then we have

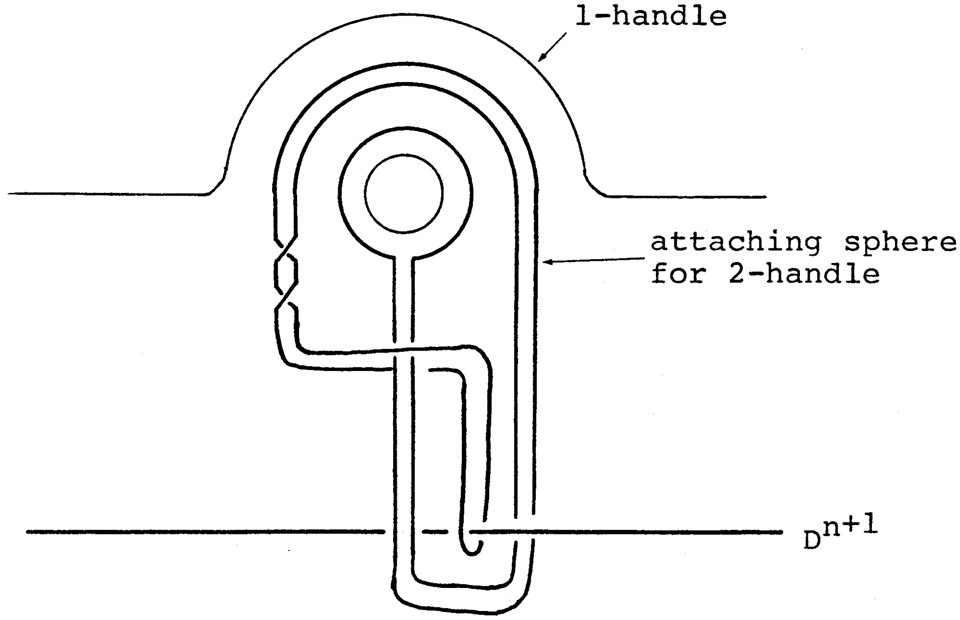
Theorem 3.1: *Any ribbon n-knot bounds an* $(n+3, n+1)$-*disk pair which has a handle presentation consisting entirely of* 1- *and* 2-*handles.*

Conversely, suppose (D^{n+3}, fD^{n+1}) is a proper disk pair having a handle presentation consisting entirely of 1- and 2-handles. Is the bounding sphere pair, $(\partial D^{n+3}, f(\partial D^{n+1}))$, a ribbon knot? We have only a partial answer to this question. Because of this, we call such a sphere pair a *weak ribbon n-knot.* Then the question becomes "Is every weak ribbon knot a ribbon knot?" In this section, we will show that the answer to the above question is yes, if we require that the attaching spheres of the 2-handles be of a special form.

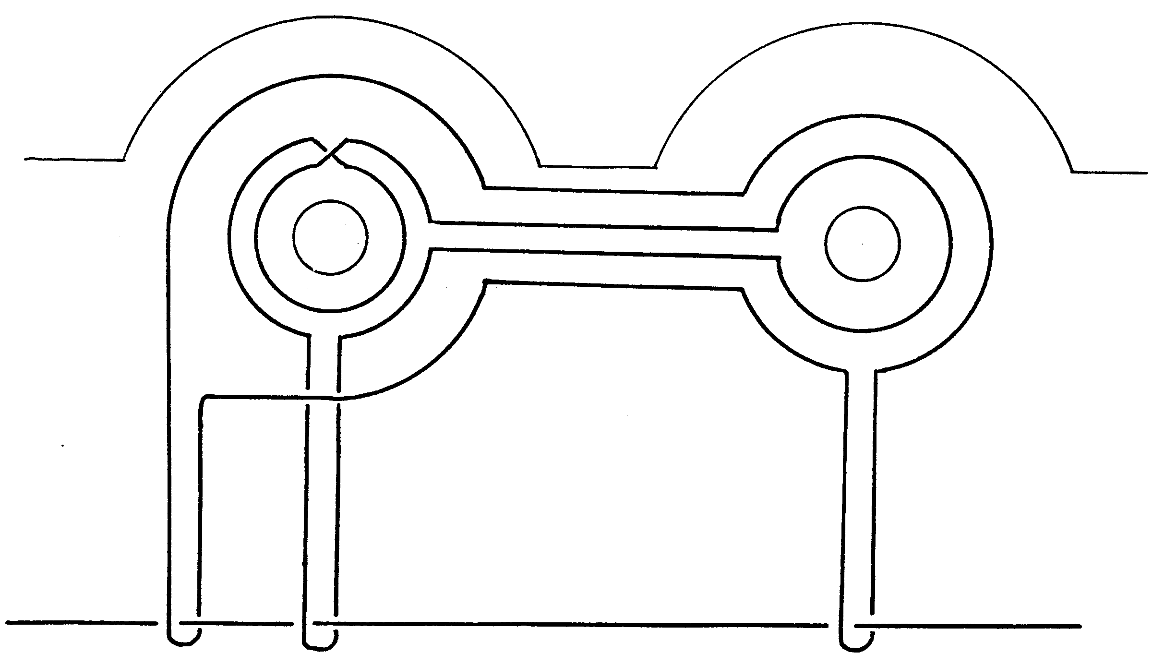
Suppose (D^{n+3}, fD^{n+1}) is a disk pair obtained by adding k 1-handles and k 2-handles to the unknotted disk pair (since the total space is a disk, there must be the same number of 1- and 2-handles). Then the exterior of fD^{n+1} in D^{n+3} has a handle decomposition built on an $(n+3)$-disk consisting of k+1 1-handles and k 2-handles. In the exterior of the bounding sphere pair, we can choose k circles, $S^1_1, \ldots, S^1_k$, such that each S^1_i goes once around the $i^{\underline{th}}$ 1-handle, and does not intersect any other 1-handle. We say that the k 1- and 2-handles are attached to (D^{n+3}, D^{n+1}) in *special cancelling pairs* if there is an isotopy of $\partial(D^{n+3} \cup$ 1-handles) which moves the attaching spheres of the 2-handles to $S^1_1 \cup \ldots \cup S^1_k$. Figure 4 shows examples and non-examples of special cancelling pair presentations.

In these schematic diagrams, the attaching spheres for the 2-handles are drawn to show their relationship to the 1-handles and the submanifold. In Figure 4(a), if n = 1, then the attaching sphere for the 2-handle is not isotopic to once around the 1-handle, and is thus not an example of a special cancelling pair presentation. In spite of this, it turns out that the total space is a disk, and the bounding sphere pair is the classical knot 9_{46}. If $n \geq 2$, then the attaching curve for the 2-handle is isotopic to once around the 1-handle, and thus is in special cancelling pair form. More generally, given any disk pair (D^{n+3}, fD^{n+1}), $n \geq 2$, obtained from the unknotted disk pair by adding a single pair of 1- and 2-handles, the presentation must be a special cancelling pair one. This is because the attaching sphere must be homotopic to once around the 1-handle, and in this range of dimensions, homotopy implies isotopy.

Figure 4(b) gives an example of a handle presentation which is not a special cancelling pair one, even in higher



(a)



(b)

Figure 4

dimensions. However, using handle moves, this presentation can be altered to become the empty presentation of the unknotted disk pair (hence in special cancelling pair form). Handle moves can also be used on the presentation in 4(a) to adjust it to special cancelling pair form.

Question: Can any 1-, 2-handle presentation of a disk pair be changed to one in special cancelling pair form by the handle moves?

These two examples, together with several others which have been checked, suggest that the answer is yes.

We now prove the promised partial converse to Theorem 3.1.

Theorem 3.2: *If a sphere pair bounds a disk pair having a handle presentation consisting entirely of* 1- *and* 2-*handles in special cancelling pair form, then the sphere pair is a ribbon knot.*

Proof: The proof follows a method of Roseman's [28], the main idea of which was originated by Sumners [34]. The idea is to use the definition of special cancelling pairs to isotope the 2-handles into cancelling position with respect to the 1-handles. This isotopy drags an $(n+1)$-disk which bounds the knot around the 1-handles. By doing some cutting and pasting on the $(n+1)$-disk, a ribbon immersed disk will be produced without altering the boundary.

So let

$$(D^{n+3}, fD^{n+1}) \cong (D^{n+3}, D^{n+1}) \cup \{h_i^1 | 1 \le i \le k\} \cup \{h_i^2 | 1 \le i \le k\},$$

where the handles are attached in special cancelling pair form. We first want to construct an $(n+1)$-disk, B, most of which is in ∂D^{n+3}, such that $\partial B = f(\partial D^{n+1})$. We will be able to get all of B in ∂D^{n+3} by introducing some ribbon singularities. To do this, we return to our usual notation: N is a regular neighborhood of fD^{n+1} in D^{n+3}; C is a collar on $W \cap N$ in W, where $W = cl(D^{n+3}-N)$. Then $N \cup C$ is an $(n+3)$-disk, and by isotopy adjustment, we can assume the 1-handles are attached in a strip homeomorphic to $D^{n+1} \times I$ in $\partial(N \cup C)$. We can then choose an $(n+1)$-disk, in $\partial(N \cup C)$, which misses the strip and whose boundary is $f(\partial D^{n+1})$. (Figure 5 illustrates this). This is the desired disk, B.

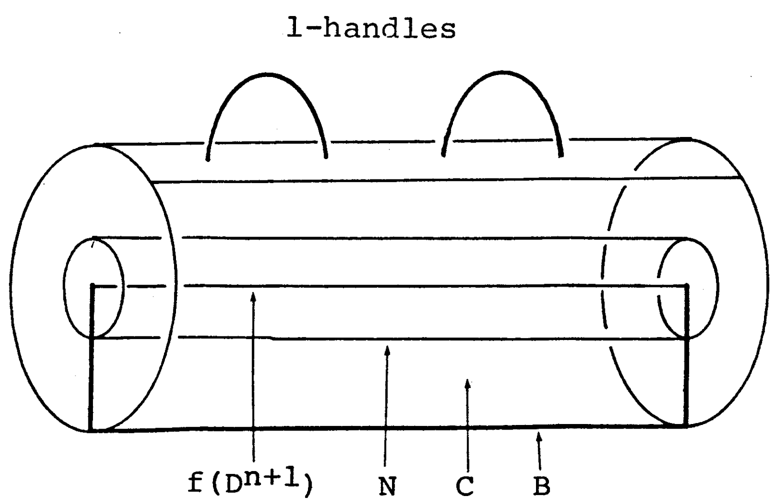


Figure 5

Without loss of generality, we may assume that the attaching sphere for each of the 2-handles intersects B transversely at a finite number of points, letting $x_1, x_2, \ldots, x_m \in B$ denote all such points. B can be expressed as

$$B = B_0 \cup B_1 \cup \ldots \cup B_m \cup P_1 \cup \ldots \cup P_m,$$

where the B_i's are disjoint, $x_i \in B_i$ for $i \geq 1$, and for each such i, P_i is a pipe from B_i to B_0 along an arc, a_i. (See Figure 6).

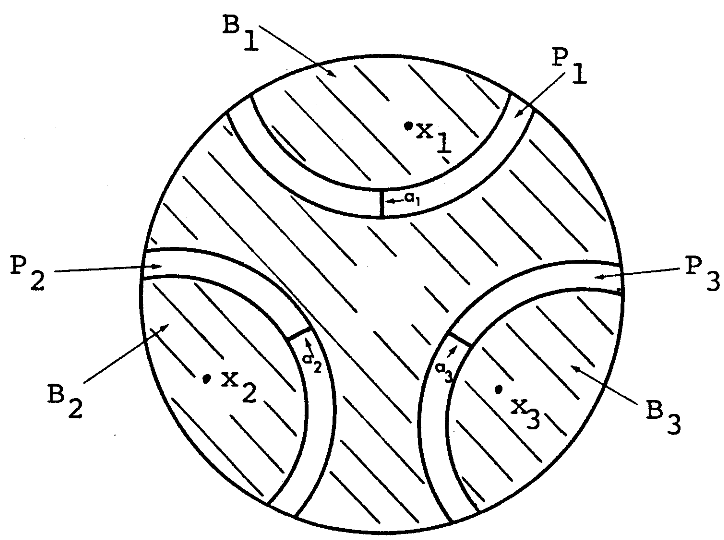


Figure 6

Each pipe, P_i, is then a 1-handle attached to $B_0 \cup B_i$, with a_i as the core of the handle. This endows each P_i with the usual 1-handle product structure, $D^1 \times D^n$.

Now, since the 1- and 2-handles attached to $N \cup C$ are in special cancelling pairs, there is an isotopy of $N \cup C \cup$ 1-handles which moves the attaching curves of the 2-handles to once around their corresponding 1-handles. Let ϕ denote the finishing homeomorphism of this isotopy.

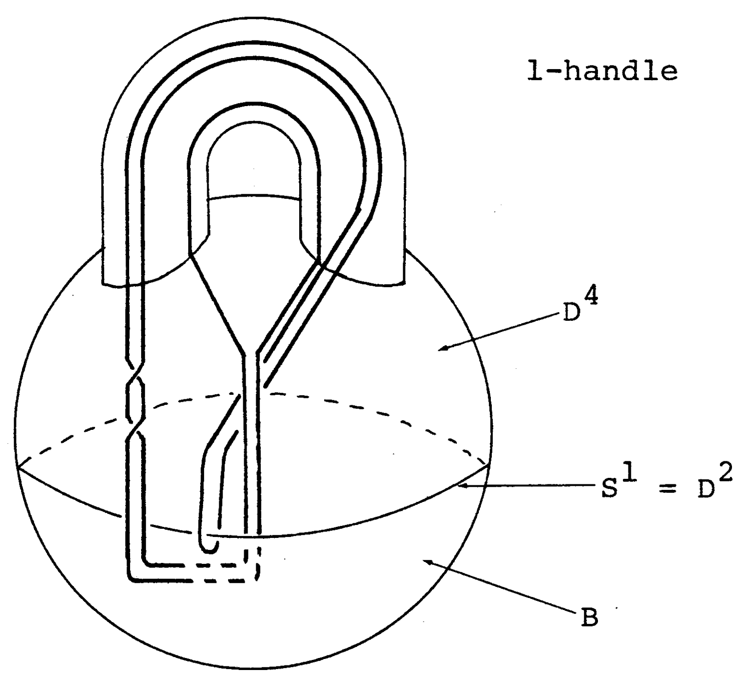
By standard transversality and regular neighborhood arguments, we can assume that each $\phi(B_i)$ $(i \geq 1)$ is contained in a distinct S^{n+1} factor in the product structure, $D^1 \times D^{n+2}$, of the 1-handles, and that $\phi(P_i)$ hits the 1-handles, h_j^1, in sets of the form $J \times D^{n+1}$, where J is a subinterval of D^1 and $D^{n+1} \subset S^{n+1}$ in the product structure, $D^1 \times D^{n+2}$. We can also assume $\phi(B_0) \subset N \cup C$ so that it does not interfere with any of the 1-handles. Finally, by replacing the 2-handles with "thinner" ones, we can suppose that $\phi(P_i) \subset \partial D^{n+3}$ for each i. The only parts of the disk, $\phi(B)$, which are not in ∂D^{n+3} then are $\phi(B_i)$ for $i \geq 1$. But this can be corrected by replacing each $\phi(B_i)$ by the set $B_i' = \mathrm{cl}(S^{n+1} - \phi(B_i))$, where S^{n+1} represents the S^{n+1} factor which contains $\phi(B_i)$ in the 1-handle. Then

$$\phi(B_0) \cup B_1' \cup \ldots \cup B_k' \cup \phi(P_1) \cup \ldots \cup \phi(P_k)$$

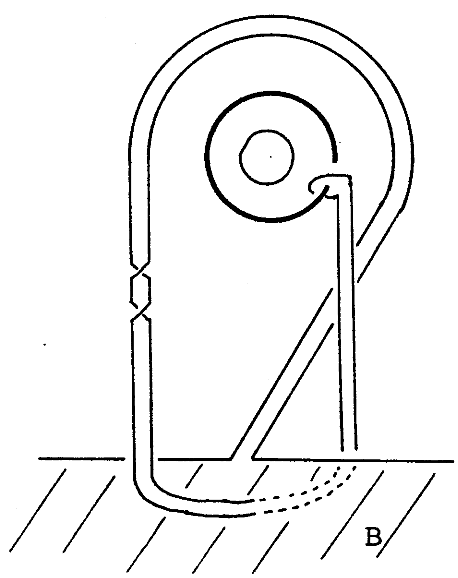
is an immersed $(n+1)$-disk contained in ∂D^{n+3}. But all the singularities are ribbon singularities by construction, since they are of the form $\phi(P_i) \cap B_j'$. $\square$

We illustrate the proof with an example. In §II.3, we obtained a handle presentation for the exterior of a disk pair which bounds the exterior of the knot 9_{46} (see Figure

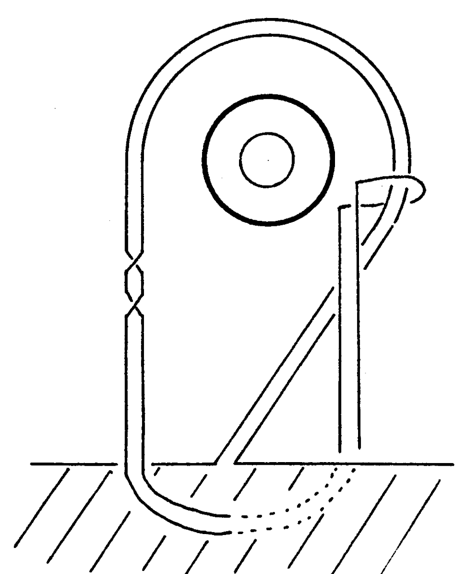
II.13). By re-inserting the neighborhood of the submanifold, we obtain a disk pair which bounds a sphere pair which represents 9_{46}. If we arbitrarily select the left 1-handle in Figure II.13 to be the 1-handle which goes around the submanifold, we obtain the handle presentation of Figure 7(a), which is a special cancelling pair one. Dragging the attaching sphere of the 2-handle back to cancelling position, we obtain (b). If this were done strictly by the method outlined in the proof, we would have pulled all three points of intersection up in the 1-handle. But noting that we can introduce a ribbon singularity in the disk by leaving the left and center intersection points fixed simplifies the calculation considerably. Replacing the small 2-disk which intersects the attaching sphere of the 2-handle by its complement in the S^3 factor, we have (c). Although we can now "see" the knot, it is very easy to make a mistake in recognizing which knot is obtained, as is pointed out by Sumners [34]. To overcome this, we prefer to pull the knot away from the 1-handle by going over the 2-handle. This can be accomplished by the indicated pipings to the unknot. Note that if the 2-handle had been attached by a different framing, we would have picked up crossovers in the band when it was dragged over the 2-handle. Once the knot is off of the 1-handle, we can cancel the 1- and 2-handles, and view the knot in the standard S^3, where it is easy to recognize.



(a)

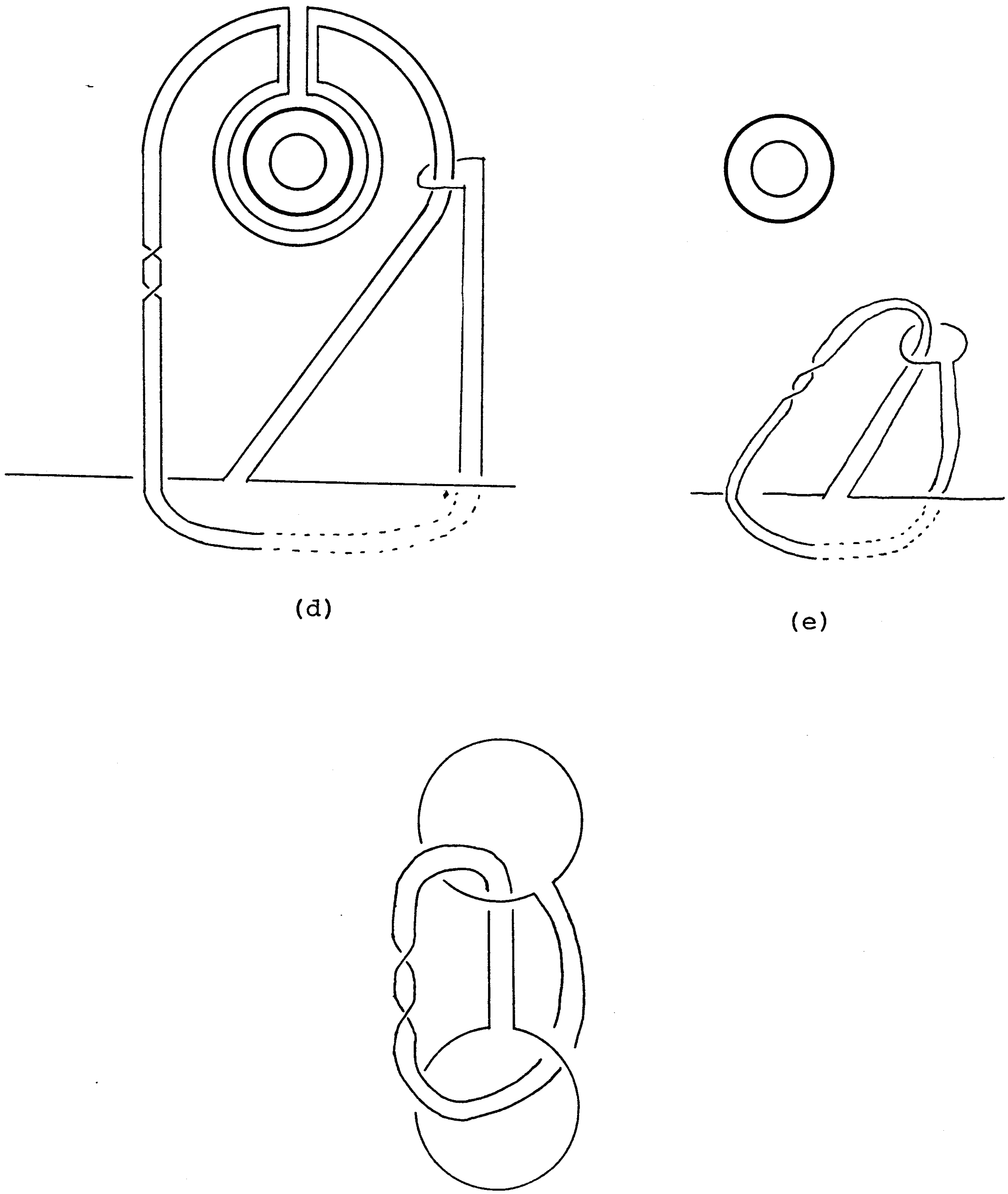


(b)



(c)

Figure 7



(d)

(e)

(f)

Figure 7

After the following definition, we will sum up, and add to, the results we have on ribbon knots. We say that a handle decomposition of a cobordism between a knot and the unknot, consisting entirely of 1- and 2-handles from the unknotted end, is in *special cancelling pair form* if the induced handle presentation of the disk pair obtained by capping off the unknotted end with the unknotted disk pair is in special cancelling pair form.

Theorem 3.3: *Let* $K = (S^{n+2}, fS^n)$ *be an n-knot. The following conditions are equivalent:*

(a) K *is a ribbon knot;*

(b) fS^n *is a fusion of a trivial n-link in* S^{n+2}*;*

(c) fS^n *is ambient isotopic to* $\partial\left[D^{n+1}\cup\{h_i^n|1\le i\le k\}\cup\{h_i^1|1\le i\le k\}\right]$ *where* $D^{n+1}\cup\{h_i^n|1\le i\le k\}$ *is contained in an equatorial* S^{n+1} *in* S^{n+2}*;*

(d) $fS^n = \partial W^{n+1}$*, where* W *is a semi-unknotted manifold in* S^{n+2}*;*

(e) fS^n *bounds a ribbon disk in* $S^{n+2}\times I$*;*

(f) K *is cobordant to the unknot by a cobordism built up with only 1- and 2-handles in special cancelling pairs from the unknotted end;*

(g) K *bounds a disk pair having a handle presentation consisting entirely of 1- and 2-handles in special cancelling pair form.*

Proof: The equivalence of (a), (b), (c), and (d) is covered in Chapter I. We will complete the proof by observing

(a) $\to$ (e) $\to$ (f) $\to$ (g) $\to$ (a).

(a) $\to$ (e): Theorem II.3.3.

(f) $\to$ (g): immediate.

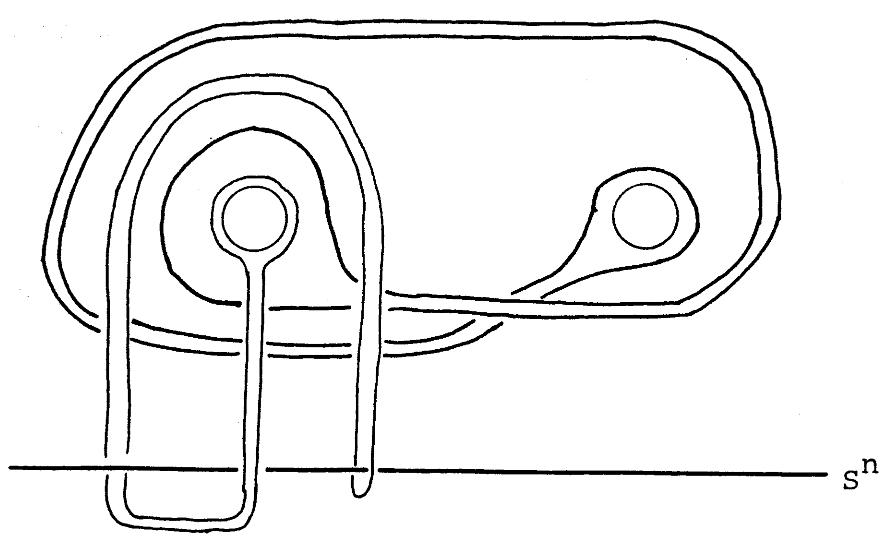
(g) $\to$ (a): Theorem 3.4.

This leaves only (e) $\to$ (f):

So suppose $fD^{n+1} \subset S^{n+2}\times I$ is a ribbon disk. From the proof of Theorem II.3.5, we know that this ribbon disk induces a cobordism between the knot $K = (f(\partial D^{n+1}), S^{n+2}\times 0)$ and the unknot having a handle decomposition consisting entirely of 1- and 2-handles from the unknotted end. The problem, of course, is that this may not be a special cancelling pair presentation. However, from the remark in §II.3, we do know the general form of the attaching spheres of the 2-handles. They all are obtained by piping two circles together, where each circle goes once around a 1-handle (possibly the 1-handle around the unknot). If each attaching curve is the result of piping a circle around the unknot to a circle around a different 1-handle, then the handle decomposition is a special cancelling pair one, and there is nothing to do. On the other hand, the handles could appear as in Figure 8(a). But this can easily be fixed by a use of the Adding Lemma (Lemma 2.4). The result of the indicated piping is as shown in Figure 8(b), since we have already seen how to compute a "push off" of the attaching curves of the 2-handles obtained using the proof of Theorem II.2.1 (see §II.3). The redrawing in Figure 8(c) makes it clear that the handle decomposition is a special cancelling pair one.

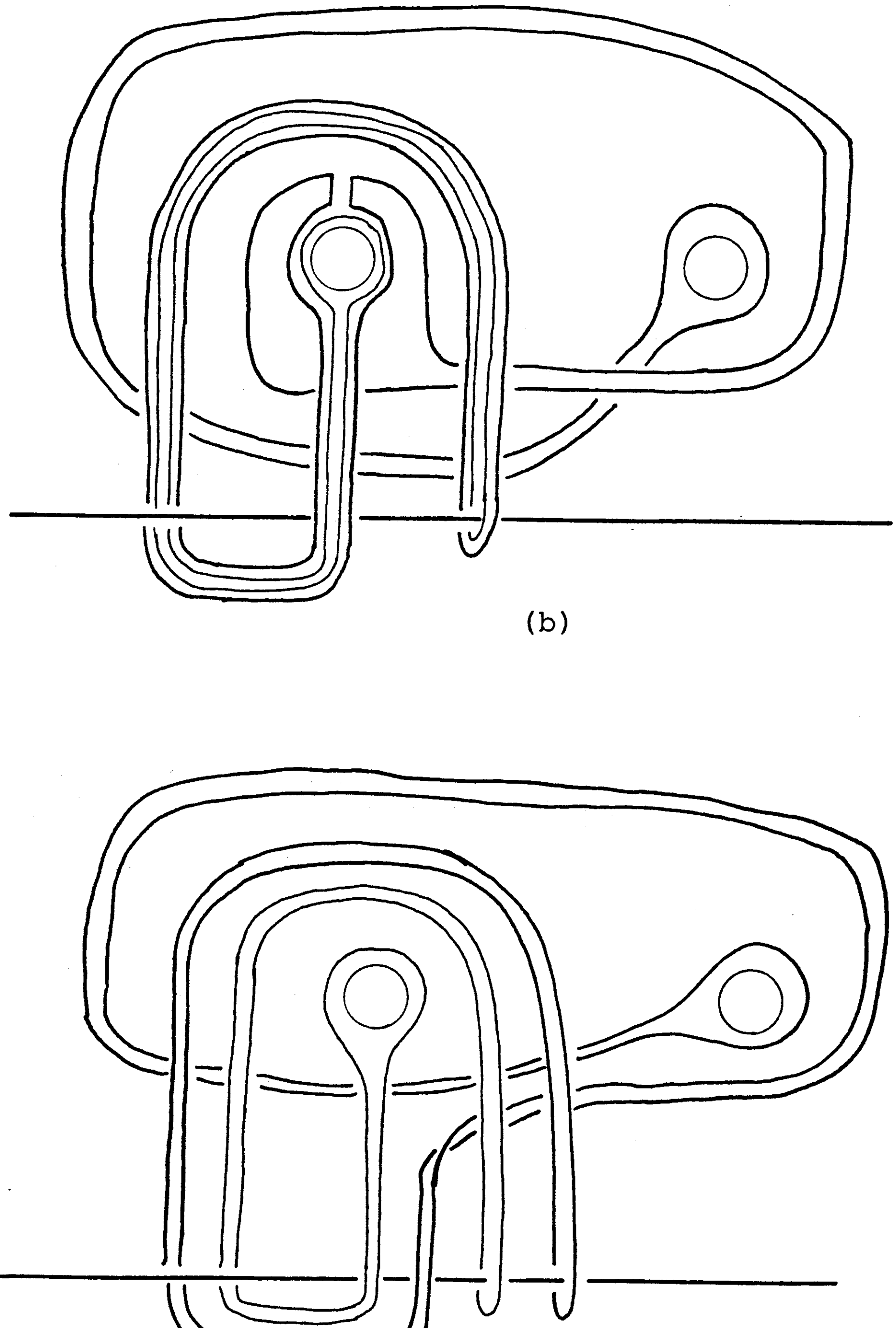
Using this technique wherever necessary, it is easy to transform any handle decomposition obtained through Theorem II.3.5 into a special cancelling pair one. This completes the proof. □

The technique just used cannot alone transform the handle decompositions in Figure 4 to special cancelling pair ones. Those require other methods, although they can be so transformed. In fact, the author knows of no 1- and 2-handle presentation for a disk pair which cannot be transformed to a special cancelling pair one using the handle moves of §2. As has been pointed out once, then, a good research problem is to investigate this to see if it can always be done.



(a)

Figure 8



(b)

(c)

Figure 8

§4. Row and Column Operations on Incidence Matrices

Our main interest here will be to apply the handle moves to disk pairs which have handle presentations with only 1- and 2-handles. Many of the techniques, however, work just as well for manifold pairs which have handle presentations with only r- and (r+1)-handles. We develop the ideas in this more general setting, specializing later when necessary. We will keep track of the effect of the handle moves using an incidence matrix.

Suppose, then, that (M^m,Q) is a compact, connected, proper manifold pair having a handle presentation of the form

$$(M,Q) \cong (N,Q) \cup \{h_i^r \mid 1 \le i \le s\} \cup \{h_i^{r+1} \mid 1 \le i \le t\}.$$

Since M and Q are both connected, we can assume

$$(4.1) \qquad r \ge 1 \quad \text{and} \quad r+1 \le m-1.$$

Let $\varepsilon(h_i^{r+1},h_j^r)$ denote the *incidence number* of the handles h_i^{r+1} and h_j^r, i.e., the intersection number of the attaching sphere of h_i^{r+1} with the belt sphere of h_i^r. Also, let A denote the $t \times s$ matrix whose $(i,j)^{\underline{th}}$ entry is $\varepsilon(h_i^{r+1},h_j^r)$.

It is well known that Lemmas 2.1 - 2.4 allow elementary row operations to be performed on the incidence matrix A (see [9], [29]). For our purposes, however, we will also need column operations. We will review the row operations before developing the column operations.

Row Operations

R1: *Any row of the incidence matrix* A *can be replaced by itself plus an integral multiple of any other row.*

To see this, we first show how to replace row i_1 by itself plus row $i_2 (i_1 \neq i_2)$ using the Adding Lemma. In the cobordism with boundary, $W = cl(M-N)$, replace $h_{i_1}^{r+1}$ by a handle whose attaching sphere is obtained by piping the attaching sphere of $h_{i_1}^{r+1}$ to a parallel of the attaching sphere of $h_{i_2}^{r+1}$ so that their orientations match. Call the (r+1)-handle so obtained $h_{i_1'}^{r+1}$. Then the formula

$$\varepsilon(h_{i_1'}^{r+1}, h_j^r) = \varepsilon(h_{i_1}^{r+1}, h_j^r) + \varepsilon(h_{1_2}^{r+1}, h_j^r)$$

holds for each j, and shows that the desired row operation has been accomplished.

To subtract row i_2 from row i_1, do the piping so the orientations of the attaching spheres disagree. Repeating the appropriate process k times, row i_1 can be replaced by itself plus or minus k times row i_2. By Remark 2.5, this row operation preserves the homeomorphism type of the pair (M,Q).

R2: *Any row of the incidence matrix can be replaced by its negative.*

This operation is easily accomplished by reversing the orientation of the appropriate (r+1)-handle.

R3: *Any two rows of the incidence matrix can be interchanged*

This is simply a matter of relabelling the (r+1)-handles.

Column Operations

With some care, the column operations which are analogous to the row operations can be derived from the row operations themselves. The idea is illustrated in the following example.

Example:

Let $A = \begin{bmatrix} 0 & 1 & -2 \\ 2 & 1 & -1 \\ 3 & -1 & -2 \end{bmatrix}$ We want to replace column 2 by itself plus column 1.

$$\begin{bmatrix} 0 & 1 & -2 \\ 2 & 1 & -1 \\ 3 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -2 & 0 \\ 2 & 1 & -1 & 0 \\ 3 & -1 & -2 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

$$\underset{\substack{R_2 \to R_2 + 2\cdot R_4 \\ R_3 \to R_3 + 3\cdot R_4}}{\overset{R_1 \to R_1 + 0\cdot R_4}{\sim}} \begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 3 & -1 & 2 \\ 0 & 2 & -2 & 3 \\ -1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$$

The first step can be easily accomplished by introducing a cancelling pair of handles so that the desired matrix results. The second step consists of performing the indicated row operations. The third step is the only problem. Figure II.14, for example, has $\begin{bmatrix} I & 0 \end{bmatrix}$ as an incidence matrix, but none of the handles can be cancelled there. In special situations, the Whitney lemma would apply to show that the desired cancellation can be done, but by attention to some preliminary detail, it can be arranged so that the cancellation can always be done to produce the desired column operation.

C1: *Any column can be replaced by itself plus any integral multiple of any other column.*

We begin by showing how to replace column j_1 by itself plus column j_2. To the given handle presentation of (M,Q), we introduce a cancelling pair of handles, h^r_{s+1} and h^{r+1}_{t+1}, so that the matrix of incidence numbers is

$$\begin{bmatrix} A & 0 \\ \alpha_j & 1 \end{bmatrix} \qquad \text{where } \alpha_j = \begin{cases} 0, \text{ if } & j \neq j_1 \text{ or } j_2, \\ 1, \text{ if } & j = j_1, \text{ and} \\ -1, \text{ if } & j = j_2. \end{cases}$$

We also require that the attaching sphere of h^{r+1}_{t+1} intersect the belt sphere of h^r_j in a single point if $j = j_1, j_2$.

By standard arguments, we may assume that the attaching curves of all the (r+1)-handles intersect $h^r_{j_2}$ in a set of the form $D^r \times x_\alpha$ for some $x_\alpha \in \partial D^{m-r}$, where the set $D^r \times D^{m-r}$ has been identified with the r-handle $h^r_{j_2}$. Since $m-r-1 \geq 1$ (follows from (4.1)), the belt sphere of $h^r_{j_2}$ has dimension greater than or equal to 1. Then we may choose an arc γ in the belt sphere of $h^r_{j_2}$ which begins at the point of intersection with the attaching sphere of h^{r+1}_{t+1}, hits all the points of intersection of the belt sphere with the attaching spheres of the other (r+1)-handles, and terminates at one of the intersection points. This arc then induces an ordering on the points of intersection $0 \times x_1, 0 \times x_2, \ldots,$ $0 \times x_q$, where $0 \times x_1$ is the intersection point on the attaching sphere of h^{r+1}_{t+1} (see Figure 9). For later reference, let n_i denote the subscript of the (r+1)-handle to which $0 \times x_i$ belongs.

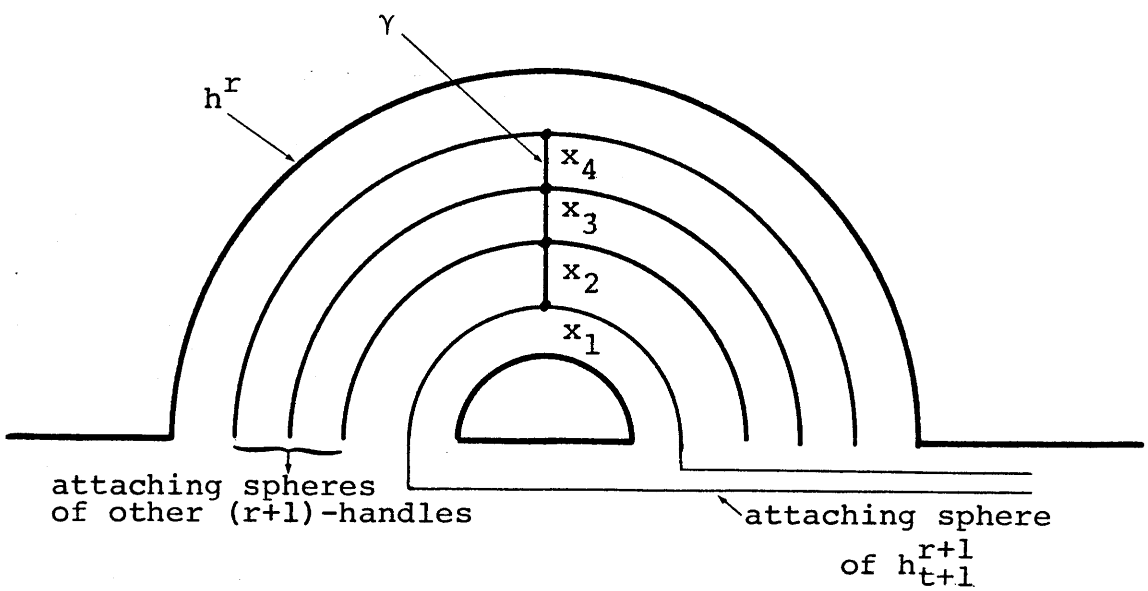


Figure 9

Let $[-\varepsilon,\varepsilon]^r$ denote the r-fold Cartesian product of $[-\varepsilon,\varepsilon]$ with itself. Then $[-\varepsilon,\varepsilon]\times\gamma$ is a pipe connecting the attaching spheres at the intersection points. We apply the Adding Lemma. Choose a collar

$$c:\partial D^{r+1}\times\partial D^{m-r-1}\times I \longrightarrow cl\left(M - (h_{t+1}^{r+1}\cup h_{n_q}^{r+1})\right)$$

on the boundary of the attaching set of h_{t+1}^{r+1}. "Add" h_{t+1}^{r+1} to $h_{n_q}^{r+1}$ by piping the attaching sphere of $h_{n_q}^{r+1}$ to $c(\partial I^{r+1}\times x_q\times 1)$ using the tube $\gamma\times[-\varepsilon,\varepsilon]^r$. Inductively, pipe the attaching sphere of $h_{n_{q-i}}^{r+1}$ to $c(\partial I^{r+1}\times x_{n_{q-i}}\times(\frac{1}{2})^i)$ using $\gamma_{q-i}\times[-\varepsilon(\frac{1}{2})^i,\varepsilon(\frac{1}{2})^i]$ where γ_{q-i} denotes the subarc of γ from x_1 to x_{q-i} (the inductive process runs from i=0 to i = q-2). See Figure 10.

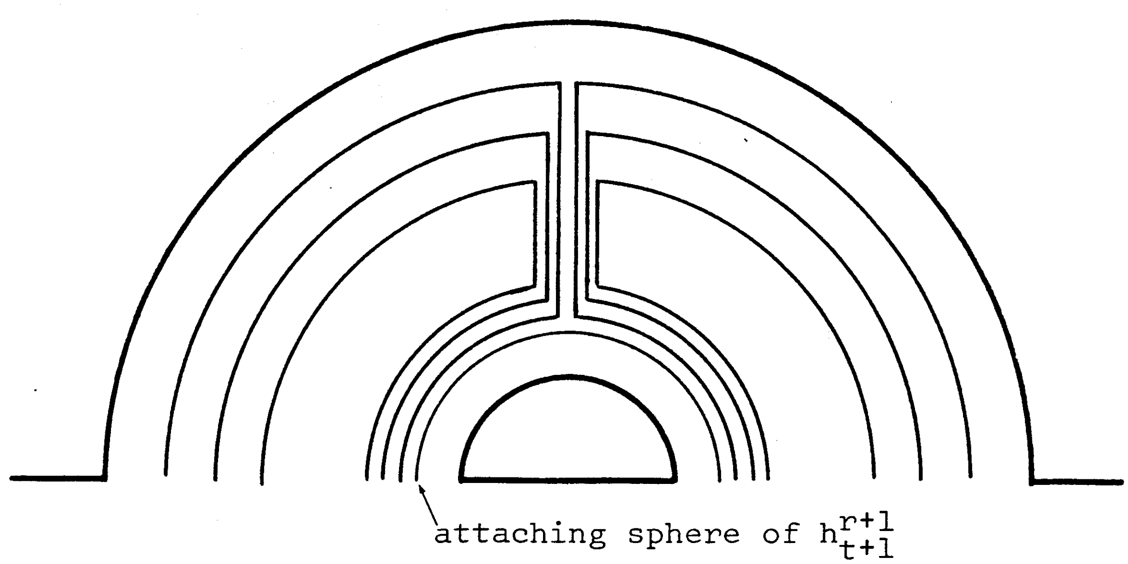


Figure 10

At each stage of the inductive process, the effect on the incidence matrix is to either add or subtract the $(n_{q-i})^{\underline{th}}$ row to (or from) the $(t+1)^{\underline{st}}$ row, depending on whether or not the piping tube matches the orientations of the attaching spheres. Thus, after the completion of the inductive procedure, the effect is that each row i $(i = 1,\ldots,n)$ has been replaced by itself plus $\varepsilon(h_i^{r+1}, h_{j_2}^r)$ times row t+1. So the $j_2^{\underline{nd}}$ column is a zero column, except for the 1 in the $(n+1)^{\underline{st}}$ row. More than that, the only point of intersection of an attaching sphere of an (r+1)-handle with the belt sphere of $h_{j_2}^r$ is $0 \times x_1$. Thus, the handles $h_{j_2}^r$ and h_{t+1}^{r+1} can be cancelled. By inserting the old $(s+1)^{\underline{st}}$ column in the $j_2^{\underline{nd}}$ slot (accomplished by relabelling h_{s+1}^r as $h_{j_2}^r$), we achieve the desired incidence matrix. This shows how to replace column j_1 by itself plus column j_2.

To replace column j_1 by itself minus column j_2, the same proof as above works if we begin with

$$\begin{bmatrix} A & 0 \\ \alpha_j & 1 \end{bmatrix} \qquad \text{where} \qquad \alpha_j = \begin{cases} 0 \text{ if } j \neq j_1, j_2 \\ -1 \text{ if } j = j_1, j_2. \end{cases}$$

By repeating these operations, it is possible to replace column j_1 by itself plus any integral multiple of column j_2.

<u>C2</u>: *Any column of the incidence matrix can be replaced by its negative.*

<u>C3</u>: *Any two columns of the incidence matrix can be interchanged.*

These follow by reorienting and relabelling.

§5. Incidence Matrices for Disk Pairs

Let (D^{n+2}, fD^n) be a disk pair, and suppose

$$(D^{n+2}, D^n) \cup \{h_i^1 \mid 1 \le i \le m\} \cup \{h_i^2 \mid 1 \le i \le m\}$$

is a handle presentation of the disk pair (since D^{n+2} is homologically trivial, there have to be the same number of 1- and 2-handles if there are no 3-handles). In this case, there is another incidence relationship not already covered by the usual incidence matrix A. This extra relationship comes from the fact that the exterior of the disk D^n, in D^{n+2}, is an (n+2)-disk with a 1-handle attached, and the incidence of the 2-handles with this 1-handle can be measured. Intuitively, these incidence numbers are just the algebraic number of times the attaching spheres of the 2-handles wind around the submanifold.

So, to measure this incidence relationship, we can replace (D^{n+2}, D^n) in the handle presentation with $(D^{n+2} \cup h_{m+1}^1 \cup h_{m+1}^2$, cocore of $h_{m+1}^2)$, where h_{m+1}^1 and h_{m+1}^2 are complementary (see Figure 11). Then the incidence numbers $\varepsilon(h_i^2, h_{m+1}^1)$, $1 \le i \le m$, give the desired information. We can then augment the $m \times m$ incidence matrix A with these additional incidence numbers as an $(m+1)$st column to obtain the $m \times (m+1)$ matrix $\tilde{A}$.

The three row operations from the previous sections apply to the augmented matrix $\tilde{A}$. However, with the column operations, we need to exercise some care. One obvious

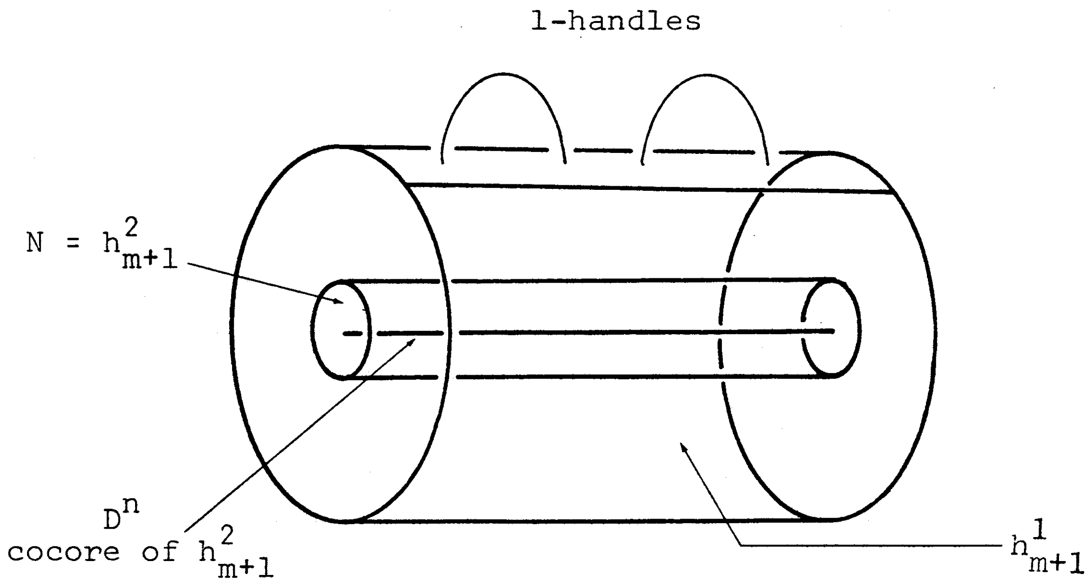


Figure 11

point would be that we would not want to interchange column m+1 with any other column, since the $(m+1)^{\underline{st}}$ column is reserved for the winding numbers of the 2-handles around the submanifold. In general, we can do any of the handle moves, as long as we are careful to keep the set $N = h^2_{m+1}$ fixed. This assures that the homeomorphism type of the disk pair is unchanged.

With this in mind, a check of the other column operations shows that we do not want to allow the $j^{\underline{th}}$ column $(1 \le j \le m)$ to be replaced by itself plus a multiple of the $(m+1)^{\underline{st}}$ column, since the handle sliding involved would move h^2_{m+1}. However, replacing the (m+1)-column by itself plus a multiple of another column can be effected leaving h^2_{m+1} fixed. Thus, for the augmented matrix, $\tilde{A}$, we have the following row and column operations which can be geometrically realized.

$$\left.\begin{array}{l}\underline{R1}:\\ \underline{R2}:\\ \underline{R3}:\end{array}\right\}\quad \text{as stated in §2.}$$

<u>C1'</u>: *Column j_1 can be replaced by itself plus any integral multiple of column j_2* $(j_2 \neq j_1, m+1)$.

<u>C2'</u>: *Any column can be replaced by its negative (same as C2).*

<u>C3'</u>: *Column j_1 can be interchanged with column j_2, provided* $j_1 \neq m+1 \neq j_2$.

We will now use the matrix operations to prove a "normal form" theorem for incidence matrices. Note that $\det(A) = \pm 1$ since the total space is a disk, hence acyclic. Using C2', we may assume that each of the first row entries of A is positive. By repeated application of the division algorithm, we can reduce the sizes of the (1,1)- and (1,2)-entries of A until one of them becomes 0, mirroring the division algorithm on the matrix level at each stage by C1'.

In this manner, we can arrange for all but one of the first row elements of A to be zero. The remaining element must be 1, since $\det(A) = \pm 1$. Repeating this for each row, and using C3', we can transform A to the $n \times n$ identity matrix, I_n. Then using C1', each element of the $(n+1)^{\underline{st}}$ column can be replaced with zero. This produces

Theorem 5.1: *Let* (D^{n+2}, fD^n) *be a disk pair having a handle presentation consisting entirely of* 1- *and* 2-*handles. Using the six handle moves* R1,R2,R3,C1',C2', *and* C3', *the handle presentation can be adjusted so that the augmented incidence matrix is of the form* $\begin{bmatrix} I & 0 \end{bmatrix}$.

In Chapter 4, we will examine some applications of this theorem.

IV. APPLICATIONS

§1. Knots Which Are Uniquely Determined By Their Exteriors

A knot, $K_1 = (S^{n+2}, f_1S^n)$, is *uniquely determined by its exterior* if, whenever the exterior of K_1 is diffeomorphic to the exterior of the knot $K_2 = (S^{n+2}, f_2S^n)$, then K_1 is equivalent to K_2. Browder [1] has shown that for $n \geq 2$, there are at most two distinct knots with diffeomorphic exteriors. Certain classes of knots are known to be uniquely determined by their exteriors (Gluck [6], Cappell [2], Levine [16], and Kearton [12]). With the aid of Theorem III.5.1, an argument of Sumners [7] yields another class of knots which are uniquely determined by their exteriors.

<u>Theorem 1.1 (Hitt and Sumners [7])</u>: *Let* (D^{n+3}, fD^{n+1}) *be a disk pair having a handle presentation composed entirely of* 1- *and* 2-*handles. If* $n \geq 2$, *then the sphere pair* $(\partial D^{n+3}, f(\partial D^{n+1}))$ *is uniquely determined by its exterior.*

The theorem is proven by constructing a diffeomorphism of pairs between the two possible sphere pairs which have the same exterior. The diffeomorphism is constructed in stages at the disk pair level: first, the diffeomorphism representing the nontrivial element of $\pi_1(SO)$ from the unknotted disk pair to unknotted disk pair is extended over the 1-handles in the handle presentation; then it is extended over the 2-handles. However, there is an obstruction to the extension over the 2-handles; namely, the mod(2) number of times the attaching spheres of the 2-handles go around the submanifold. But Theorem III.5.1 shows that for any such given disk pair, we can change the handle decomposition such that the obstruction vanishes, thus allowing the diffeomorphism of pairs to be completed.

Rephrasing Theorem 1.1 in our terminology, we have

Corollary 5.2: *If* $n \geq 2$, *any weak ribbon n-knot is uniquely determined by its exterior.*

Proof: Immediate, from the definition of weak ribbon knot.□

Corollary 5.3: *For* $n \geq 2$, *ribbon n-knots are uniquely determined by their exteriors.*

Proof: Ribbon n-knots are weak ribbon n-knots.□

§2. Another Proof of Theorem III.3.2

Theorem III.5.1 can also be used to give a different proof of Theorem III.3.2. Instead of constructing a ribbon immersed disk which bounds the knot, a semi-unknotted manifold which bounds the knot can be constructed by a method of Omae [24]. We will sketch the proof for the $(D^{n+3},D^{n+1}) \cup h^1 \cup h^2$ case. For more than a single pair of such handles, the proof is completely analogous.

Let B be the oriented (n+1)-disk constructed in the proof of Theorem III.3.2 such that $\partial B = \partial D^{n+1}$ and $B \subset \partial D^{n+3}$. As before, we may assume the attaching sphere of h^2 intersects B transversely at a finite number of points. Each point of intersection has an incidence number, either +1 or -1. Now, by Theorem III.5.1, we may assume (without loss of generality) that the sum of the incidence numbers is zero. So, there must be an even number of such points, say $x_1,x_2,\ldots,x_{2k}$.

Suppose, for the moment, that there are only two points of intersection. Then we can form a manifold, M, in $\partial[D^{n+3} \cup h^1 \cup h^2]$ by piping B to itself along the subarc x_1x_2 of the attaching sphere of h^2. M is orientable, since the incidence numbers at x_1 and x_2 disagree. The boundary of the piping tube is a disjoint union of two n-spheres. These spheres are unlinked in $\partial[D^{n+3} \cup h^1 \cup h^2]$. This is because the attaching sphere of h^2 is isotopic to once around h^1, and disjoint discs in $\partial[D^{n+2} \cup h^1 \cup h^2]$ can be obtained which bound the spheres as in the proof of Theorem III.3.2. This shows that M has a trivial system of n-spheres, hence is semi-unknotted.

Inductively, assume the piping can be done to obtain a semi-unknotted manifold in $\partial[D^{n+3} \cup h^1 \cup h^2]$ whenever there are 2m-2 intersection points with total incidence zero. Suppose that there are 2m intersection points with total incidence zero in a disk pair. The orientation of the attaching sphere of h^2 induces a cyclic ordering on the points. We may choose two adjacent points whose incidence numbers disagree, and pipe these together so that the pipe passes through no other intersection points. This leaves 2m-2 points whose total incidence is zero, and they can be piped together by assumption. So if we use a thick pipe between the first two points, and successively thinner pipes for the other 2k-2 points, the two can be done together without interfering. Again, a check shows the resulting manifold is semi-unknotted (see Figure 1).

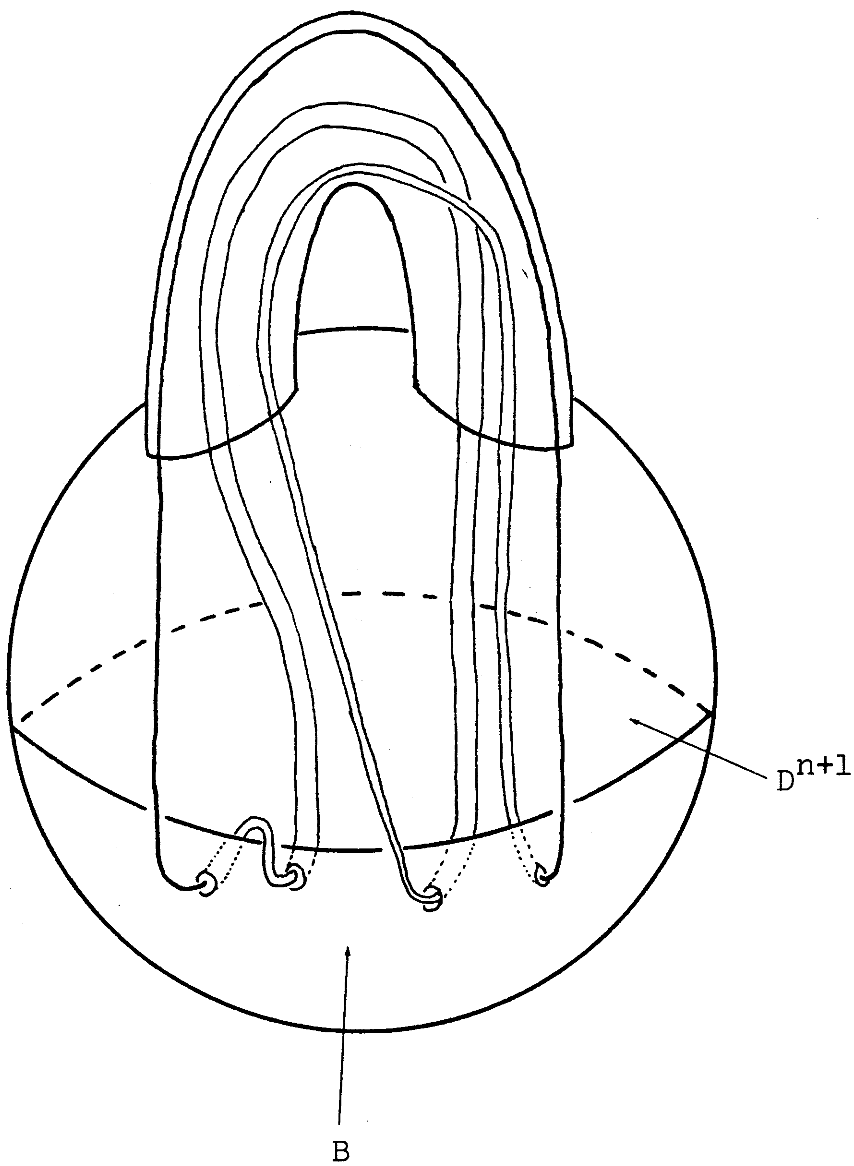


Figure 1

§3. An Example

The purpose of this section is to apply the handle calculus, developed in Chapter III, to modify a given handle presentation. In Chapter II, we calculated a handle presentation for a disk pair which bounds the knot 9_{46} (see Figure II.13). Sumners [34] also has a handle presentation for a disk pair which bounds 9_{46} (see Figure II.14). Using the handle calculus, we will show that the two disk pairs are diffeomorphic.

We begin with the Sumners presentation in Figure 3(a). The framing, or product structure, used on the 2-handle by Sumners to obtain 9_{46} on the boundary is such that if the attaching sphere of the 2-handle is ambiently isotoped to once around the 1-handle, a push-off of the attaching sphere appears as indicated in Figure 2.

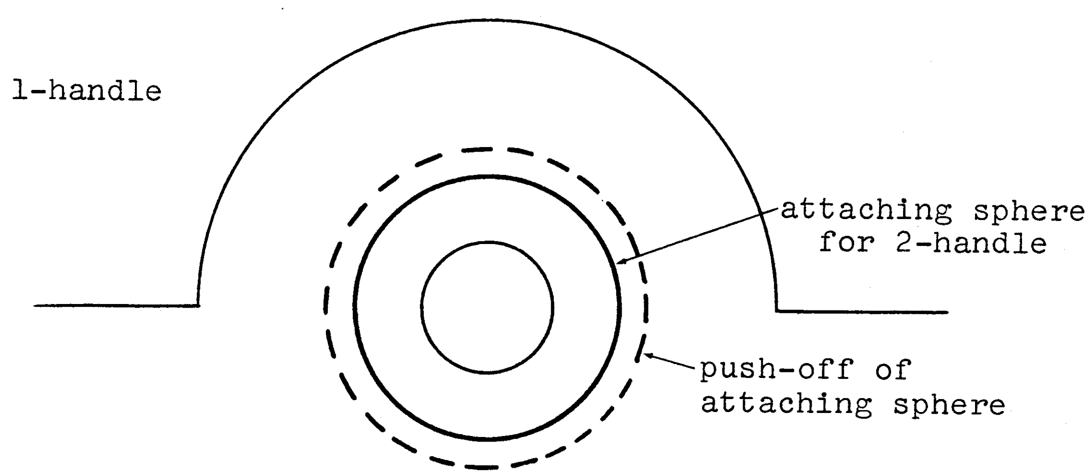


Figure 2

In Figure 3(b), a cancelling pair of 1-, 2-handles has been added. Here, there are infinite cyclic many different framings which could be used, but we choose the one indicated in Figure 2.

If we slide the right 1-handle over the left 1-handle, we obtain (c). The idea is to use the Adding Lemma until we are in a position to cancel the original two handles. First, we replace the original 2-handle, by itself plus the new one. Again, there are infinite cyclic many ways to frame the piping arc. We choose the one illustrated in (d), which is obtained using the product structure on the 1-handle, as in §III.4. Figure (e) is a re-drawing of (d).

We want to eventually cancel the original 1- and 2-handles. This can be accomplished by the methods used in §III.4 for the column operations, and is indicated in (f). Care must be exercised to ensure that the correct push-offs are used. Then the original 1- and 2-handles are in cancelling position, and we cancel them in (g). Re-drawing (g), we obtain (h), which is the handle presentation obtained in Chapter II for the knot 9_{46}.

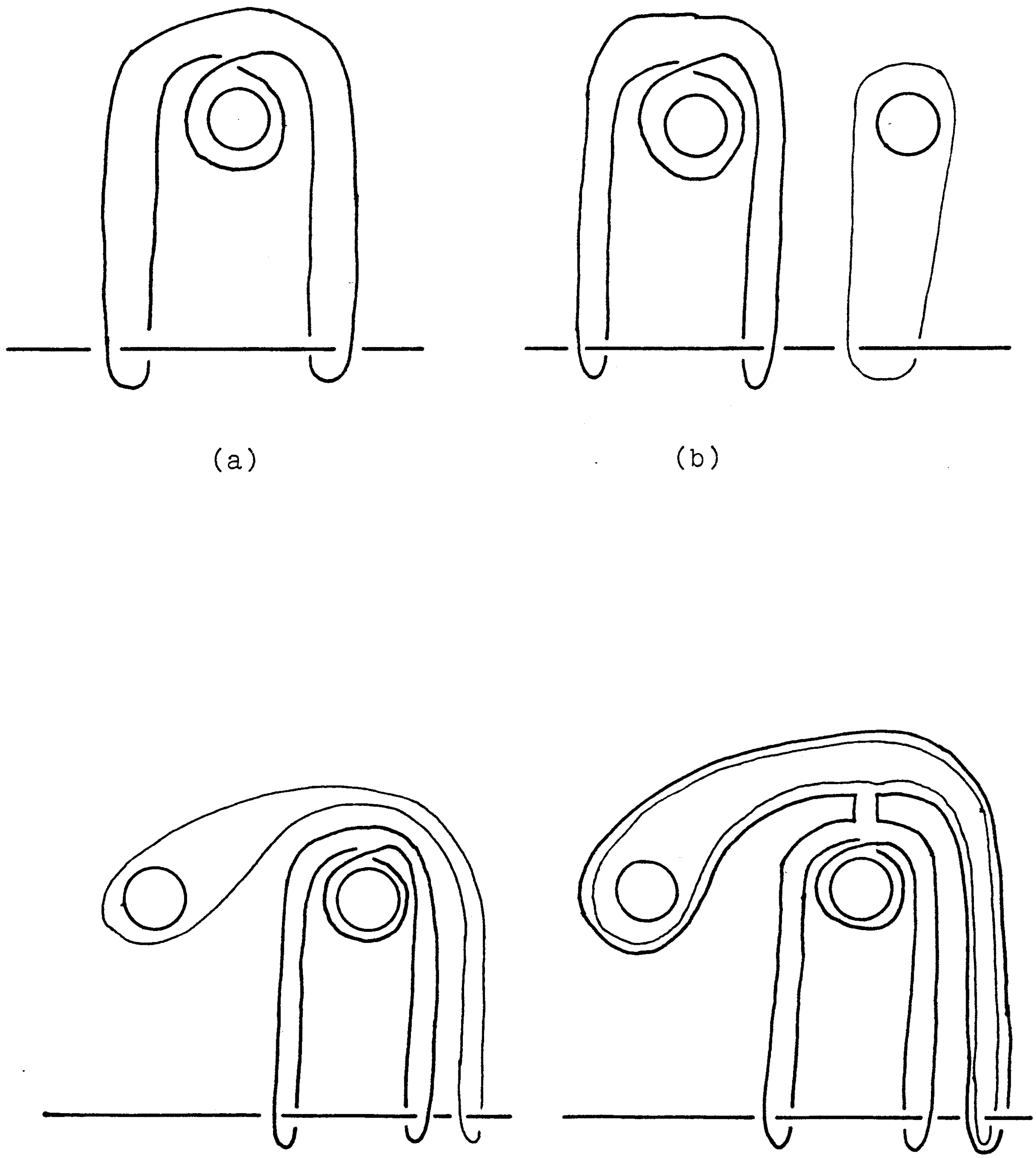
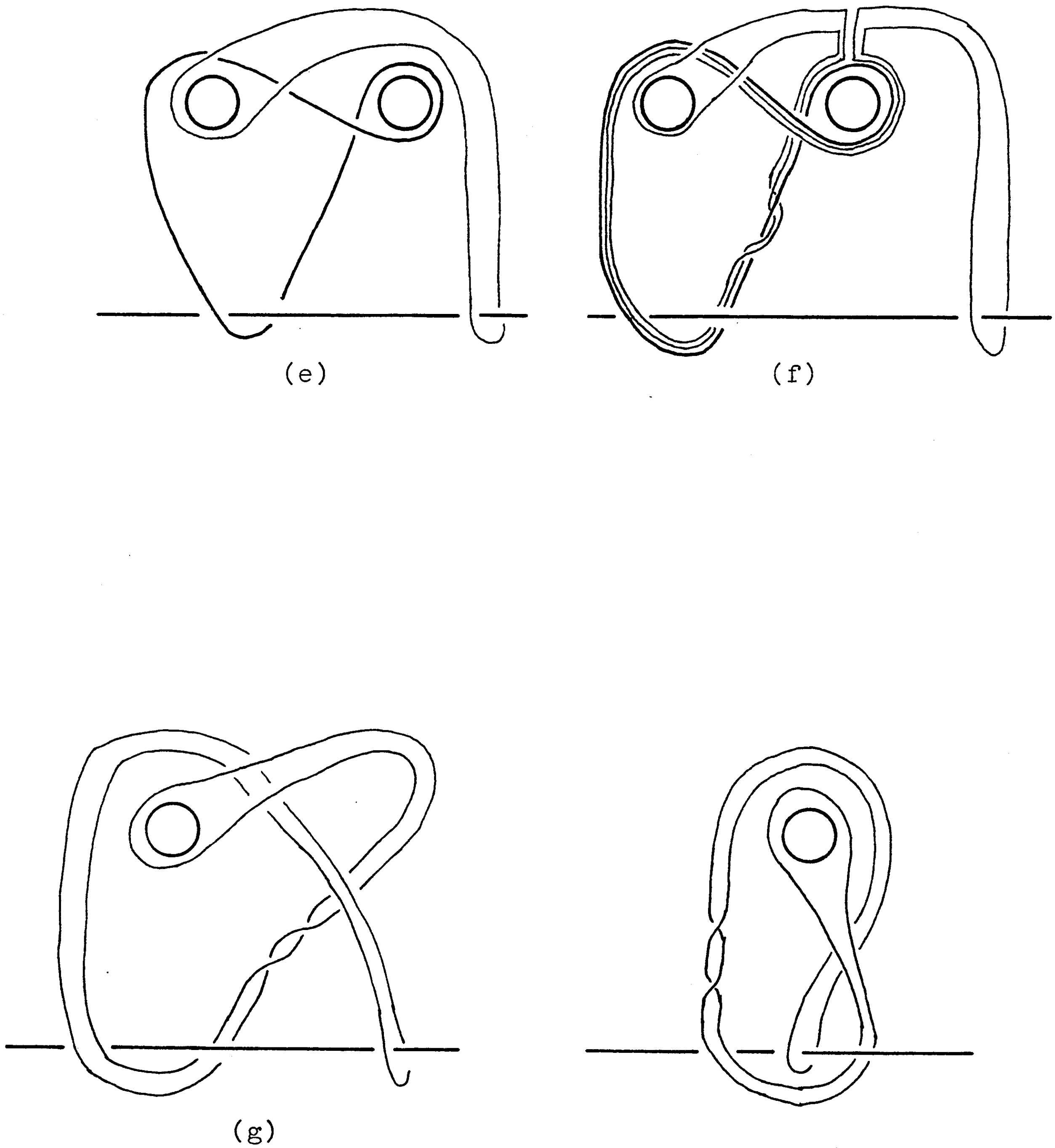


Figure 3



(e) (f) (g) (h)

Figure 3

§4. Homological Torsion in the Infinite Cyclic Coverings of Knot Exteriors.

The purpose of this section is to examine the ramifications of torsion in a homology group of the infinite cyclic covering of a slice knot on the handle structure of a null-cobordism of the slice knot. This is done in Theorem 4.3. An application of the theorem is given in §5.

Let A be an R-module. We say $x \in A$ is a *torsion element* (or, has *R-torsion*), if there is some $r \in R$, $r \neq 0$, such that $r \cdot x = 0$. If $x \in A$ is not a torsion element, it is *torsion free* (*R-torsion free*). Also, if each element of A is a torsion element, A is called a *torsion* R-module; and, if each element of A is torsion free, A is called *torsion free*.

To facilitate the proof of the theorem, we use the following lemma, the proof of which is a generalization of an argument due to Sumners [33].

Lemma 4.1: *Let* K *be a finite CW complex of dimension n, and suppose there is an epimorphism* $\phi : \pi_1(K) \longrightarrow Z$. *Let* $\tilde{K}$ *denote the infinite cyclic covering of* K *corresponding to* ϕ.

(i) If $H_n(K;Q) = 0$, *then* $H_n(\tilde{K};Z) = 0$; *also,*

(ii) If $H_n(\tilde{K};Z) = 0 = H_{n-1}(K;Z)$, *then* $H_{n-1}(K;\tilde{Z})$ *is Z-torsion free.*

Proof: (i) Let t be a generator of the infinite cyclic group of covering transformations, and let $\Gamma = Q[t,t^{-1}]$ (the rational group ring of the group of covering transformations). Following Milnor [21] and Levine [17], the rational chain groups $C_q(\tilde{K};Q)$ are then finitely generated over Γ, the generators being in one-to-one correspondence with the q-cells of K. Since Γ is Noetherian, the homology $H_*(\tilde{K};Q)$ is a finitely generated Γ module. But in fact, Γ is a principal ideal domain, so we have the decomposition theory of finitely generated modules over principal ideal domains to aid in the analysis.

As in Milnor [21], consider the short exact sequence of Γ-modules

$$0 \longrightarrow C_*(\tilde{K};Q) \xrightarrow{(t-1)} C_*(\tilde{K};Q) \longrightarrow C_*(K;Q) \longrightarrow 0$$

where the homomorphism $(t-1)$ is Γ-module multiplication. This gives rise to a long exact sequence of homology, part of which is

$$0 \longrightarrow H_n(\tilde{K};Q) \xrightarrow{(t-1)_n} H_n(\tilde{K};Q) \longrightarrow H_n(K;Q) \longrightarrow \cdots$$

By assumption, $H_n(K;Q) = 0$, so $(t-1)_n$ is an epimorphism (in fact, an isomorphism). But then $t-1 \in \Gamma$ divides any element in $H_n(\tilde{K};Q)$. By the theory of finitely generated modules over a principal ideal domain, $H_n(\tilde{K};Q)$ is a direct sum of cyclic Γ-modules of the form $\Gamma/(a)$, where (a) is the ideal generated by $a \in \Gamma$. Since $t-1$ divides each element of $H_n(\tilde{K};Q)$, none of the direct summands can be free. Thus $H_n(\tilde{K};Q)$ is a torsion Γ-module.

On the other hand, since $C_{n+1}(\tilde{K};Q) = 0$, $H_n(\tilde{K};Q) = Z_n(\tilde{K};Q)$, the module of n-cycles. And since Γ is a principal ideal domain, $Z_n(K;Q)$ is free, being a submodule of the free

Γ-module $C_n(\tilde{K};Q)$. Thus $H_n(\tilde{K};Q)$ is free over Γ. This forces $H_n(\tilde{K};Q) = 0$, since it is both free and torsion.

By the universal coefficient theorem, we have $H_n(\tilde{K};Q) \cong H_n(\tilde{K};Z) \otimes_Z Q \oplus \mathrm{Tor}(H_{n-1}(\tilde{K};Z), Q)$, so that $0 = H_n(\tilde{K};Z) \otimes_Z Q$. Thus $H_n(\tilde{K};Z)$ is a torsion Z-module. But as before, $H_n(\tilde{K};Z)$ is also a free Z-module, since

$$H_n(\tilde{K};Z) = Z_n(\tilde{K};Z) \subset C_{n+1}(\tilde{K};Z).$$

Thus $H_{n+1}(\tilde{K};Z) = 0$.

(ii) Let $\Lambda = Z[t,t^{-1}]$, the integral group ring of the covering transformation group, and let T_{n-1} denote the Λ-submodule of $H_{n-1}(\tilde{K};Z)$ consisting of all Z-torsion elements (since all the chain and homology groups in this part of the proof are with Z coefficients, we will dispense with the coefficient designation from here on).

As in part (i), we have the short exact sequence of Λ-modules

$$0 \longrightarrow C_*(\tilde{K}) \xrightarrow{(t-1)} C_*(\tilde{K}) \longrightarrow C_*(K) \longrightarrow 0$$

inducing a long exact sequence on homology:

$$\cdots \longrightarrow H_{n-1}(\tilde{K}) \xrightarrow{(t-1)_{n-1}} H_{n-1}(\tilde{K}) \longrightarrow H_{n-1}(K) \longrightarrow \cdots .$$

By assumption, $H_{n-1}(K) = 0$, so $(t-1)_{n-1}$ is an epimorphism. Lemma II.8 in Kervaire [14] says that whenever $(t-1)_q$ is an epimorphism, T_q is a finite group. So T_{n-1} is a finite Λ-module. Now, multiplication by t induces an automorphism on the finite group T_{n-1}, so there is some $p > 0$ such that t^p is the identity automorphism.

Let K_p^n denote the p-fold cyclic covering of K. From Shinohara and Sumners [31], there is a long exact sequence relating the homology groups of K_p to those of $\tilde{K}$,

$$\cdots \longrightarrow H_n(\tilde{K}) \longrightarrow H_n(K_p) \xrightarrow{\partial} H_{n-1}(\tilde{K}) \xrightarrow{(t^p-1)_{n-1}} H_{n-1}(\tilde{K}) \longrightarrow H_{n-1}(K_p) \longrightarrow \cdots$$

Now, $T_{n-1} \subset \ker(t^p-1)_{n-1} = \mathrm{im}(\partial)$. But $H_n(\tilde{K}) = 0$ by assumption, so $H_n(K_p) \cong \mathrm{im}(\partial)$. Thus T_{n-1} is embedded as a submodule of $H_n(K_p)$. Since K_p has dimension n though, $H_n(K_p)$ is a free Z-module. Thus $T_{n-1} = 0$, i.e., $H_{n-1}(\tilde{K})$ is Z-torsion free. □

Let (S^{n+2}, fS^n) be a knot, and X the exterior of the knot. Since X is a compact (n+2)-manifold with non-empty boundary, it can be collapsed away from the boundary onto a finite (n+1)-complex K. Let $\tilde{K}$ denote the infinite cyclic covering of K corresponding to the Hurewicz homomorphism $\phi:\pi_1(K) \quad H_1(K)$ (=Z). Since K is a homology S^1, $H_{n+1}(K;Q) = 0 = H_n(K;Z)$ if $n>1$, so the preceeding lemma applies to prove the following corollary. For $n=1$, an analogous argument can still be used to give the

<u>Corollary 4.2</u>: *If X is the exterior of an n-knot* $(n\geq 1)$, *and* $\tilde{X}$ *its infinite cyclic cover, then* $H_{n+1}(\tilde{X};Z) = 0$ *and* $H_n(\tilde{X};Z)$ *is Z-torsion free.*

Using Lemma 4.1, we can prove the main theorem.

Theorem 4.3: *Let* $K = (S^{n+2}, fS^n)$ *be a slice knot,* $(S^{n+2}\times I, w^{n+1})$ *a cobordism from K to the unknot,* $\nu(w)$ *a tubular neighborhood of* w *in* $S^{n+2}\times I$, *and* $W = cl(S^{n+2}\times I-\nu(w))$. *Also, let* X *denote the exterior of* K, *and* $\tilde{X}$ *its infinite cyclic covering. If* $H_q(\tilde{X})$ *has torsion then any handle decomposition of* W *from the unknotted end has either a* $(q+2)$- *or an* $(n-q+2)$-*handle.*

Proof: We prove the contrapositive of the theorem. Assume the cobordism is parameterized so that the 0-level is the unknotted end, and let $U = W \cap (S^{n+2}\times 0)$, the exterior of the unknot. Without loss of gererality, we may assume $X = W \cap (S^{n+1}\times 1)$. Suppose the negation of the conclusion, i.e., that there is some handle decompositon of W of the form

$$W = U\times I \cup \{h^i_j \mid 1 \le i \le n+2,\ i \ne q+2,\ i \ne n-q+2,\ j\in A_i\}$$

where for each i, A_i is a finite indexing set.

Since handles are simply connected, they lift to any covering space to induce a handle structure, so

$$\tilde{W} = \tilde{U}\times I \cup \{h^i_\alpha \mid i \ne q+2,\ i \ne n-q+2,\ \alpha\in B_i\}$$

where for each i, B_i is an infinite indexing set disjoint from A_i. Let

$$W_{q+1} = U\times I \cup \{h^i \mid i \le q+1\} \quad \text{and}$$

$$\tilde{W}_{q+1} = \tilde{U}\times I \cup \{h^i_\alpha \mid i \le q+1\}$$

The contrapositive will be proven by applying lemma 4.1 to

W_{q+1} to show $H_q(\tilde{W}_{q+1})$ is torsion free, and deducing $H_q(\tilde{X})$ is torsion free from that.

To apply Lemma 4.1, note that W_{q+1} is homotopy equivalent to a finite (q+1)-dimensional CW complex, say L. Furthermore, since W is obtained from W_{q+1} by adding handles of index greater than q+2, we have $H_m(W) = H_m(W_{q+1})$ for $m \leq q+1$, the isomorphism being inclusion induced. But W is a homology S^1, and $W_{q+1} \simeq L$, so $H_*(S^1) \cong H_*(L)$. By the universal coefficient theorem, $H_*(S^1;R) \cong H_*(L;R)$ for any ring R. So Lemma 4.1 applies to show that $H_q(\tilde{L})$ is torsion free, whence $H_q(W_{q+1})$ is also. But $H_q(\tilde{W}_{q+1}) = H_q(\tilde{W})$ (same reason as in the base space W), so $H_q(\tilde{W})$ is torsion free.

Consider now the dual decomposition of $\tilde{W}$,

$$\tilde{W} = \tilde{X}\times I \cup \{\overset{*}{h}{}^i_\alpha \mid i \neq q+1,\ n-q+1\},$$

where $\overset{*}{h}{}^i_\alpha$ is the dual handle of h^i_α. Since $\tilde{W}$ is obtained from $\tilde{X}\times I$ by attaching handles of index other than q+1, $H_{q+1}(\tilde{W},\tilde{X}\times I) = 0$. The long exact sequence of the pair $(\tilde{W},\tilde{X}\times I)$

$$\cdots \longrightarrow H_{q+1}(\tilde{W},\tilde{X}\times I) \xrightarrow{\partial} H_q(\tilde{X}\times I) \xrightarrow{i_*} H_q(\tilde{W}) \longrightarrow \cdots$$

then shows that the inclusion induced homorphism

$$i_*: H_q(\tilde{X}\times I) \longrightarrow H_q(\tilde{W})$$

is a monomorphism. Thus, $H_q(\tilde{X}\times I)$ must also be torsion free. But $\tilde{X}\times I$ is homotopy equivalent to $\tilde{X}$, so $H_q(\tilde{X})$ is torsion free, as desired. □

In the next section, we examine an application of this corollary.

§5. Slice Knots

In 1962, Fox [5] posed the

Question: Is every (classical) slice knot a ribbon knot?

We will examine the question in higher dimensions.

Yajima [39] has shown that Example 12 in [3] is an example of a slice 2-knot which is not a "simply knotted 2-sphere". Since Yanagawa [40] has shown that the definition of simply knotted 2-sphere is equivalent to that of ribbon 2-knot, this then is an example of a 2-knot which is slice, but not ribbon. We will construct examples here of n-knots which are slice, but not ribbon, for each $n \geq 2$. The 2-twist-spun trefoil will be used for these examples (see Zeeman [43]), although Example 12 of Fox [3] works just as well. In fact, R. Litherland reportedly has shown that the two knots are equivalent.

Let $K = (S^{n+2}, fS^n)$ be an n-knot. Choose $x \in fS^n$. Then x has an unknotted disk pair neighborhood (B_x^{n+2}, B_x^n). The *disk pair associated with the knot* K is

$$(S^{n+2}, fS^n) - \mathrm{int}(B_x^{n+2}, B_x^n).$$

To *p-spin* a knot, one p-spins its associated disk pair, which is defined as follows: given a disk pair (D^{n+2}, gD^n), its *p-spin* is the sphere pair

$$(S^{n+p+2}, \sigma_p(g)(S^{n+p})) = \partial((D^{n+2}, gD^n \times D^{p+1})).$$

1-spinning a knot can be thought of as taking the Cartesian product of the associated disk pair with S^1, and capping off the top and bottom with the unknotted disk pair. In this sense, then, the associated ball pair is being "spun" around S^1. To *n-twist-spin* a knot, the associated ball pair is twisted n full times during the 1-spinning (see Zeeman [43], for a rigorous definition).

Let X denote the exterior of the 2-twist-spun trefoil, and $\tilde{X}$ its infinite cyclic cover. Zeeman [43] shows that the 2-twist-spun trefoil is a fibered knot, with fiber the lens space $L(3,2)$. Since $\pi_1(L(3,2)) \cong Z_3$, it follows that $\pi_1(\tilde{X}) \cong Z_3$, so $H_1(\tilde{X}) \cong Z_3$. Since any even-dimensional knot is slice (Kervaire [14]), we can apply Corollary 4.4 to see that any handle decomposition of any null-cobordism of the 2-twist-spun trefoil must have a 3-handle from the unknotted end. By Theorem II.3.5, then, the 2-twist-spun trefoil is a slice knot which is not a ribbon knot.

For dimensions larger than 2, we use the p-spin of the 2-twist-spun trefoil ($p \geq 1$). We are aided by a theorem of Sumners.

<u>Theorem 5.1 (Sumners [35])</u>: *Let* X *denote the exterior of a knot,* K, *and* Y *the exterior of the p-spin of* K. *Then*

$$H_i(\tilde{Y}) \underset{\Lambda}{\cong} \begin{cases} H_i(\tilde{X}) \text{ if } i \leq p \\ H_i(\tilde{X}) \oplus H_{i-p}(\tilde{X}) \quad \text{if } i > p. \end{cases}$$

Taking K to be the 2-twist-spun trefoil, then, for any $p>1$, we have

$$\begin{aligned} H_1(\tilde{Y}) &\cong H_1(\tilde{X}) \\ &\cong Z_3. \end{aligned}$$

Theorem 4.3 implies that any handle decomposition of any null-cobordism of the p-spin of the 2-twist-spun trefoil has either a 3-handle or a (2+p-1+2)-handle from the unknotted end, i.e., a 3-handle or a (p+3)-handle. Again, Theorem II.3.5 shows that the knot cannot be a ribbon knot. This proves the

Theorem 5.2: *For each* $n\geq 2$, *there is an n-knot which is a slice knot, but not a ribbon knot.*

§6. An Unknotting Theorem

In this section, an unknotting theorem for ribbon knots is proven. An unknotting theorem is a theorem which gives necessary and sufficient conditions for a sphere pair (S^{n+2},kS^n) to be unknotted. Papakyriakopoulos [25] in the case n=1, Shaneson [30] in the case n=3, Levine [16] in the case n≥4, and Stallings [32], have the following:

<u>Theorem 6.1</u>: *Let* (S^{n+2},fS^n) *be a sphere pair for* $n\neq 2$. *Then* (S^{n+2},fS^n) *is unknotted if and only if* $S^{n+2}-fS^n \simeq S^1$. *In fact,* (S^{n+2},fS^n) *is unknotted if and only if* $\pi_i(S^{n+2}-fS^n) \cong \pi_i(S^1)$ *for* $i \leq [\frac{1}{2}(n+1)]$.

We will show that a ribbon n-knot (S^{n+2},fS^n), $n\neq 2$, is unknotted if and only if $\pi_1(S^{n+2}-fS^n) \cong Z$. Yanagawa [42, Theorem 2.2] has this result for n=2, but there is a gap in the proof of Lemma 2.4, as pointed out by Suzuki in [36]. At this time, the case n=2 still appears to be open.

<u>Theorem 6.2 (Unknotting Theorem for Ribbon Knots)</u>: *If* $K = (S^{n+2},fS^n)$ *is a weak ribbon knot,* $n\neq 2$, *and if* X *is the exterior of* K, *then* K *is unknotted if and only if* $\pi_1(X) \cong Z$.

<u>Proof</u>: If K is unknotted, then $X \simeq S^1$, so $\pi_1(X) \cong Z$. Conversely suppose $\pi_1(X) \cong Z$. If n=1, the conclusion is a special case of the more general theorem of

Papakyriakopoulos [25], so assume $n \geq 3$. Since K is a weak ribbon knot, there is a cobordism $(S^{n+2} \times I, w)$, with exterior W, between K and the unknot such that

$$W = U \times I \cup \{h_i^1 \mid 1 \leq i \leq m\} \cup \{h_i^2 \mid 1 \leq i \leq m\},$$

where U is the exterior of the unknot. Thus, W is homotopy equivalent to a 2-dimensional CW-complex. Dually, we have

$$W = X \times I \cup \{h_i^{n+1} \mid 1 \leq i \leq m\} \cup \{h_i^{n+2} \mid 1 \leq i \leq m\},$$

so that $\pi_i(X) \cong \pi_i(W)$ for $i<n$. Thus $\pi_1(W) \cong \pi_1(X) \cong Z$, since $n \geq 3$. This shows that the infinite cyclic cover, $\tilde{W}$, of W is simply connected. Now, W is a homology S^1, so Lemma 3.1(i) implies that $H_2(\tilde{W}) = 0$. But since $\tilde{W}$ is two-dimensional, $H_i(\tilde{W}) = 0$ for $i \geq 1$, so $\tilde{W}$ is contractible. Thus $\pi_i(W) \cong \pi_i(S^1)$ for each i, and hence $W \simeq S^1$. The conclusion now follows from the Shaneson-Levine unknotting theorem.□

As a corollary to the proof, we have

Corollary 6.3: *If* Y *is a homology* S^1 *which is homotopy equivalent to a finite* 2*-dimensional* CW *complex, and if* $\pi_1(Y) \cong Z$, then $Y \simeq S^1$.

Assume the hypotheses of the above theorem. If in the proof of the theorem, we cap off the cobordism $(S^{n+2}I, w)$ at the unknotted end with the unknotted disk pair, we obtain a disk pair $(D^{n+3}, f'D^{n+1})$ with the following properties:

(i) $n+1 \geq 4$;

(ii) the exterior, E, of the disk pair is homotopy equivalent to S^1; and

(iii) $\pi_1(\partial E) \cong \pi_1(E)$.

Van Kampen's theorem shows that $\pi_1(E) \cong Z$, so (ii) follows from the corollary. Also, $\pi_1(\partial E) = \pi_1(X) \cong Z$, so (iii) is true.

We want to apply the following theorem due to Kato [11, Cor. 4.7]:

Theorem 6.4: *Assume* $n \geq 4$. *Then an n-disk pair* (D^{n+2}, gD^n) *is unknotted if the exterior, E, is of the same homotopy type as* S^1, *and if* $\pi_1(\partial E) \cong \pi_1(E)$.

Thus, the disk pair $(D^{n+3}, f'D^{n+1})$ is unknotted, which shows that the cobordism $(S^{n+2}\times I, w)$ in the proof of Theorem 5.2 is the product cobordism. This proves the

Corollary 6.5: *Let* $n \geq 3$. *If* $K = (S^{n+2}, fS^n)$ *is a weak ribbon knot such that* π_1*(exterior of* K) $\cong Z$, *then any null-cobordism of* K *built up with only* 1- *and* 2-*handles from the unknotted end is the product cobordism.*

Corollary 6.6: *Let* $n \geq 3$. *If* (D^{n+3}, fD^{n+1}) *is a disk pair having a handle presentation consisting of only* 1- *and* 2-*handles, and if* $\pi_1(D^{n+3}-fD^{n+1}) \cong Z$, *then* (D^{n+3}, fD^{n+1}) *is the unknotted disk pair.*

Appendix: RESEARCH PROBLEMS

For the sake of convenience, we have collected several of the questions raised in the dissertation, and augmented the list with some related problems.

1. (a) From the definitions on page 11, is there a definition 1 ribbon knot which is not a definition 2 ribbon knot? In particular, is the 2-twist-spun trefoil a definition 1 ribbon knot? If it does bound an immersed disk in S^4, what are the singularities?

(b) Given any slice n-knot, it bounds an (n+1)-disk in $S^{n+2}\times I$. Can this disk be pushed down into $S^{n+2}\times 0$ to yield an immersed disk which bounds the knot? If so, perhaps a hierarchy of slice knots could be established, depending on the types of singularities of the immersed disk. This hierarchy might translate into a heirarchy of handle decompositions of null-cobordisms of these slice knots, as was the case for ribbon knots in this dissertation.

2. (a) Is every Seifert manifold of a ribbon n-knot ambient isotopic to one in the form of Theorem I.3.1 (page 25)?

(b) Given a classical ribbon knot, is there a Seifert surface in the form of Theorem I.3.1 whose genus is minimal among all Seifert surfaces?

3. Following the definition of semi-unknotted manifolds on page 26, it was remarked that for any semi-unknotted manifold $M^{n+1} \subset S^{n+2}$, $M \cong \#(S^1 \times S^n) - B$, where B is an embedded (n+1)-disk in $\#(S^1 \times S^n)$.

(a) Is there an (n+1)-manifold of this form in S^{n+2} which is not semi-unknotted?

(b) Is there a non-ribbon n-knot which has a Seifert manifold diffeomorphic to $\#(S^n \times S^1) - B$?

4. Concerning the Sumners handle presentation for the exterior of the classical knot 9_{46} (Figure II.14), if the tubular neighborhood of the submanifold is re-inserted with a "twist" using a non-trivial element of $\pi_1(SO(2))$, it is possible to obtain homotopy 4-disks like the Mazur manifold. With the handle presentation shown in Figure II.13, when the submanifold is re-inserted, it is not clear if the resulting 4-manifold is a disk or not. Perhaps a handle presentation of a disk pair can be found so that when the submanifold is replaced with a twist, a different disk pair results. This would produce 2 classical knots with the same complement.

5. Given a ribbon n-knot which is a fusion of a trivial link of m components, there are m ways to construct a cobordism to the unknot using the method of §II.3, depending

on which n-sphere plays the role of the unknot. Are these cobordisms distinct?

6. (a) Given different presentations of the same ribbon knot as fusions of trivial links, are the ribbon disk pairs constructed using the methods of §II.3 different?

(b) Is it possible to have different ribbon disks which bound equivalent knots? All the ribbon disks which bound the unknot of dimension n ($n \geq 3$) are equivalent by Corollary IV.6.6.

7. (a) Can any handle presentation of a disk pair, consisting entirely of 1- and 2-handles, be changed to a special cancelling pair one using the handle moves (see page 62)?

(b) If a knot bounds such a disk pair, is it necessarily a ribbon knot? A negative answer here would show that there are slice knots which are not ribbon knots.

(c) A classical knot, $K = (S^3, fS^1)$, is a *homotopy ribbon knot* if there is a cobordism $(S^3 \times I, w)$ of K to the unknot such that $\pi_1(S^3 - fS^1) \twoheadrightarrow \pi_1(S^3 \times I - w)$ (the inclusion induced homomorphism) is onto. We then have the following set containments for classical knots:

$$\{\text{ribbon}\} \subset \{\text{weak ribbon}\} \subset \{\text{homotopy ribbon}\} \subset \{\text{slice}\} .$$

Which inclusions, if any, are proper?

8. From Theorem IV.4.3, we know that given a slice n-knot with q-dimensional homological torsion in the infinite

cyclic covering of its exterior, any handle decomposition of any null cobordism of the knot has either a (q+2)- or an (n-q+2)-handle from the unknotted end. Are there examples where there is a (q+2)-handle but not an (n-q+2)-handle, and where there is an (n-q+2)-handle but not a (q+2)-handle?

9. A simple knot is a knot $K = (S^{2n+1}, fS^{2n-1})$ such that $\pi_i(\text{exterior of } K) \cong \pi_i(S^1)$ for $i \leq n$. For a given simple knot, K, let A be a Seifert matrix for K. Kervaire [14] shows how to construct a simple (2n-1)-knot, K', with A as its Seifert matrix. In the construction,

$$K' = \partial(D^{2n} \cup \{h_i^n \leq 1 \leq i \leq 2_j\}).$$

Levine [16] has shown that higher dimensional simple knots are classified by their Seifert matrices, so K and K' are equivalent. Is there a way to construct a cobordism from K' to the unknot from the Seifert manifold $D^{2n} \cup \{h_i^n\}$ in the same way a null cobrodism was constructed in Chapter II from the Seifert manifolds of §I.3? If so, what is the handle structure of such a cobordism?

10. Kato [11] shows that there are two distinct disk pairs which have the same exterior. Are there more than two distinct disk pairs which have the same exterior?

11. Given a disk pair (D^{n+3}, fD^{n+1}) having a handle presentation consisting of 1- and 2-handles, for n=1 it is

known that π_1(exterior) does not classify the pair. For example, Sumners [34] gives two disk pairs which have the same groups, but which bound distinct knots. These examples are obtained by changing the framing on the attaching set of a 2-handle in a handle presentation. What happens in higher dimensions? Is the framing a red-herring, or are different disk pairs produced there, too? Can higher dimensional (weak) ribbon knots be classified by π_1?

12. Can the handle structure of the exterior of a knot be related to the handle structure of the exterior of the p-spin of the knot? To the n-twist-spin of the knot?

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